Exercises, week 7

1 Covariant derivative commutator

By definition, the commutator of covariant derivatives acting on a contra-variant vector defines the Riemann tensor

$$[\nabla_\alpha, \nabla_\beta]\, V^\lambda = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\lambda \equiv R^\lambda{}_{\rho\beta\alpha} V^\rho \,. \tag{1}$$

Show the equivalent relationship for a covariant tensor, i.e.

$$[\nabla_\alpha, \nabla_\beta]\, T_{\lambda\kappa} = \quad ? \tag{2}$$

This relation also holds for the metric. If the covariant derivative is consistent with the metric ($\nabla_\lambda g_{\mu\nu} = 0$) it implies an algebraic constraint for $R_{\mu\nu\kappa\lambda}$. What is this constraint? Notice that we derived this constraint before but using a completely different approach.

2 Riemann tensor of a flat metric

Consider the Minkowski metric

$$ds^2 = dx^\mu dx^\nu \, \eta_{\mu\nu} \,, \tag{3}$$

but written in a different coordinate system. The metric in this system is

$$g_{\alpha\beta}(y) = \frac{dx^\mu}{dy^\alpha} \frac{dx^\nu}{dy^\beta} \, \eta_{\mu\nu} \,, \tag{4}$$

The Riemann tensor in this new coordinate system is given by the usual expression

$$\begin{aligned} R_{\mu\nu\kappa\lambda} &= \frac{1}{2} \left(\frac{\partial^2 g_{\mu\kappa}}{\partial y^\nu \partial y^\lambda} + 3 \text{ permutations} \right) \\ &\quad + g_{\sigma\rho} \Gamma^\sigma{}_{\mu\kappa} \Gamma^\rho{}_{\nu\lambda} - g_{\sigma\rho} \Gamma^\sigma{}_{\mu\lambda} \Gamma^\rho{}_{\nu\kappa} \end{aligned} \tag{5}$$

This has to vanish, since the original metric was flat. Show this explicitly!

As intermediate steps, you should show that for this special case the Christoffel symbol in the new system reads ($g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$)

$$\Gamma^\mu_{\alpha\beta} = g^{\mu\nu} \frac{d^2 x^\kappa}{dy^\alpha dy^\beta} \frac{dx^\lambda}{dy^\nu} \eta_{\kappa\lambda} \tag{6}$$

and

$$\begin{aligned} \frac{\partial^2 g_{\mu\kappa}}{\partial y^\nu \partial y^\lambda} &= \frac{d^2 x^\rho}{dy^\mu dy^\lambda} \frac{d^2 x^\sigma}{dy^\nu dy^\kappa} \eta_{\rho\sigma} \\ &\quad + \text{terms that vanish after symmetrization} \end{aligned} \tag{7}$$

3 Riemann tensor of an embedded manifold

We do the same as in the last problem, but this time we construct an embedded manifold. In particluar, the index on the new coordinate y^α runs over less dimensions than the index on the old coordinates x^μ.

In any case, the embedded metric is

$$\gamma_{\alpha\beta}(y) = \frac{dx^\mu}{dy^\alpha}\frac{dx^\nu}{dy^\beta}\,\eta_{\mu\nu}\,, \tag{8}$$

that formally looks the same equation as (4). [To simplify life, you might consider the metric of the original space $\eta_{\mu\nu}$ to have Euclidean signature in the following.]

Obviously, the induced Riemann tensor (= the Riemann tensor using the induced metric (8)) does not vanish in general and the proof in the last problem must fail somewhere.

It turns out, that the culprit is the inverse of the induced metric that shows up in the Christoffel symbol. Consider the expression ($\gamma^{\mu\nu}$ is the inverse of $\gamma_{\mu\nu}$)

$$P^\mu_{\ \nu} = \gamma^{\alpha\beta}\frac{dx^\mu}{dy^\alpha}\frac{dx^\kappa}{dy^\beta}\eta_{\kappa\nu} \tag{9}$$

Show that for the coordinate transformation in the last problem one has $P^\mu_{\ \nu} = \delta^\mu_{\ \nu}$.

For the embedding on the other hand $P^\mu_{\ \nu}$ is a projection operator, in the sense that

$$P^\mu_{\ \nu}P^\nu_{\ \kappa} = P^\mu_{\ \kappa} \tag{10}$$

and $P^\mu_{\ \mu}$ is the dimension of the embedding. Show these two relations. Also express the Riemann tensor of the embedding in terms of the projection operator.