Exercises, week 5

1 Covariant curl

Show that the covariant curl actually is equal the ordinary curl for a symmetric affine connection

$$\nabla_\mu V_\nu - \nabla_\nu V_\mu = \partial_\mu V_\nu - \partial_\nu V_\mu \tag{1}$$

2 Covariant divergence

Show that the covariant divergence can be rewritten in the following form

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} V^\mu) \,. \tag{2}$$

For this relation, you have to show

$$\Gamma^\mu{}_{\mu\lambda} = \frac{1}{2} g^{\mu\nu} \frac{dg_{\mu\nu}}{dx^\lambda} = \frac{1}{\sqrt{g}} \partial_\lambda (\sqrt{g}). \tag{3}$$

Remember that for a symmetric matrix M we have

$$\operatorname{tr} \partial \log M = \operatorname{tr} M^{-1} \partial M \tag{4}$$

and

$$\log \det M = \operatorname{tr} \log M \,. \tag{5}$$

For a bonus, consider how (4) would generalize without the trace.

3 Levi-Civita 1

We have seen that the following two expressions are tensors in GR

$$\epsilon^{\mu\nu\kappa\lambda} / \sqrt{g} \tag{6}$$

and

$$\epsilon_{\mu\nu\kappa\lambda} / \sqrt{g} \equiv \epsilon^{\alpha\beta\gamma\delta} g_{\alpha\mu} g_{\beta\nu} g_{\gamma\kappa} g_{\delta\lambda} / \sqrt{g} \tag{7}$$

Calculate the following contraction

$$\epsilon^{\mu\nu\kappa\lambda} \epsilon_{\mu\nu\kappa\alpha} / g = -6 \delta^\lambda_\alpha \tag{8}$$

4 Levi-Civita 2

We want to calculate

$$\nabla_\alpha(\epsilon^{\mu\nu\kappa\lambda}/\sqrt{g})\,. \tag{9}$$

This is suspected to vanish, since in the free falling frame, the first derivatives of $g_{\mu\nu}$ vanish and the covariant derivative becomes the ordinary derivative. Show this explicitly by using the fact that $\epsilon/\sqrt{g}$ is a tensor and the definition of the corresponding covariant derivative (using the results of problems 2 and 3 above).