Exercises, week 11

1 Quadrupole radiation in different directions

We have seen that the power radiated from a quadrupole in a certain solid angle is

$$\frac{dP}{d\Omega} = \frac{G\omega^6}{4\pi}\Lambda_{ij,lm}D^*_{ij}(\omega)D_{lm}(\omega) \tag{1}$$

where $D(\omega)$ is a collection of Fourier modes of the quadrupole moment. The direction of the solid angle is hereby encoded in Λ

$$\Lambda_{ij,lm} = P_{il}P_{jm} - \frac{1}{2}P_{ij}P_{lm} \tag{2}$$

where P is the projection into the plane orthogonal to the solid angle. Now consider the quadrupole of a planet moving around a very massive object in the xy-plane

$$D_{ij}(\omega = 2\omega_p) = mR^2 \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \exp(2i\omega_p t) \tag{3}$$

What is the radiation in the z-direction? What is the radiation in the x-direction?

2 Multipoles in ED

In electrodynamics, the radiation of a distribution of charges can be calculated using the multi-pole expansion. The monopole and dipole is hereby given by

$$M = \int d^3x\rho(x)\,, \quad D^i = \int d^3x\, x^i\, \rho(x)\,, \tag{4}$$

where ρ is the charge distribution. One curious fact about this construction is that the dipole in principle depends on the chosen coordinate system. How does the dipole change when we shift the coordinate system $\vec{x} \to \vec{x} + \vec{a}$. Why is this ambiguity usually not a problem when the radiation is calculated?

3 Multipoles in GR

We have seen that the dipoles depend on the chosen coordinate system. In ED this is not an an issue since one could argue that the monopole has to vanish for an antenna. In GR however, the monopole and dipoles don't vanish (the energy density is positive unlike the charge density).

$$M = \int d^3x T^{00}(x)\,, \tag{5}$$

$$D_i = \int d^3x\, x^i\, T^{00}(x)\,, \tag{6}$$

$$Q_{ij} = \int d^3x\, x^i\, x^j\, T^{00}(x)\,, \tag{7}$$

So is the produced radiation ambiguous and depends on the coordinate system used?

The solution to the conundrum is that only Fourier modes with nonvanishing frequency ω produce radiation. Show that the monopole is time-independent by using energy-momentum conservation. Also show that the dipole is at most linear in time.