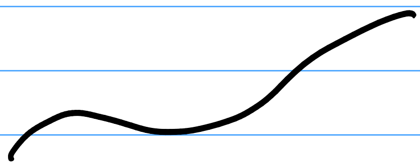
Applications:

Covariant derivative along a curve:

Remember that the ordinary derivative of a covariant vector field $\partial_\mu A^\nu(x)$ transforms non-trivially.

The same is true if we consider a vector along a curve:

$$A^\mu(\tau) \qquad \left(x^\mu(\tau)\right)$$



$$A'^\mu(\tau) = \frac{\partial x'^\mu}{\partial x^\nu}(\tau) A^\nu(\tau)$$

$$\left(\frac{d}{d\tau}A(\tau)\right)' = \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha}\frac{\partial x^\alpha}{\partial \tau} A^\nu + \frac{\partial x'^\mu}{\partial x^\nu}\frac{d}{d\tau}A^\nu(\tau)$$

Also in this case, we can define a covariant derivative:

$$\frac{D}{D\tau} A^\mu = \frac{\partial}{\partial \tau} A^\mu + \Gamma^\mu{}_{\nu\alpha} A^\alpha \frac{\partial x^\nu}{\partial \tau}$$

and likewise for covariant fields.

In case the vector is defined in the full space and not just along the path, this construction reduces to the normal convariant derivative in the sense:

$$\frac{D}{D\tau} A^\mu = \frac{\partial x^\kappa}{\partial \tau} D_\kappa A^\mu$$

$$= \frac{\partial x^\kappa}{\partial \tau} \partial_\kappa A^\mu + \frac{\partial x^\kappa}{\partial \tau} \Gamma^\mu{}_{\kappa\lambda} A^\lambda$$

Notice that it is not always possible to extend a vector along a path to a vector field in all space time.

More specifically, for a particle in motion with four-velocity

$$u^\mu = \frac{dx^\mu}{\partial \tau}$$

the geodesic equation reads in this notation:

$$\frac{D}{D\tau} u^\mu = f^\mu \quad \text{(external force)}$$

$$\frac{D}{D\tau} u^\mu = \frac{d^2 x^\mu}{(d\tau)^2} + \Gamma^\mu_{\lambda\nu} \frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau}$$

Electrodynamics:

The Maxwell's equations in SR read:

$$\partial_\alpha F^{\alpha\beta} = -j^\beta$$

$$\epsilon^{\alpha\mu\nu\lambda} \partial_\alpha F_{\mu\nu} = 0$$

$$\Leftrightarrow \quad \partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu} + \partial_\mu F_{\nu\alpha} = 0$$

$$(\Leftrightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$$

In order to write this in GR, we have to use the covariant derivative and the metric

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$$

$$D_\mu F^{\mu\nu} = -j^\nu$$

Notice that

$$D_\kappa A_\nu - D_\nu A_\kappa = \partial_\kappa A_\nu - \partial_\nu A_\kappa$$

Also, the Bianchi identity will reduce to the one with ordinary derivatives (see exercises).

Still, the Maxwell equation will contain the Christoffel symbol and the metric and describe how the electromagentic fields behave in gravity.

The electromagentic force acting on a particle can be written:

$$\begin{aligned} f^\alpha &= e F^\alpha{}_\beta \frac{dx^\beta}{d\tau} \\ &= e\, g^{\alpha\kappa} F_{\kappa\beta} \frac{dx^\beta}{d\tau} \end{aligned}$$

which is already a contravariant vector.

Particle (charge) current

In SR, the current was given as

$$\boxed{SR} \qquad J^\alpha = \sum_n e_n \int d\tau \, \delta^4(x^\mu - x_n^\mu) \frac{\partial x_n^\mu}{\partial \tau}$$

The delta function here has to be a scalar density

$$\int d^4x \, \delta(x-y) \, \overset{\text{scalar}}{\Phi(x)} = \overset{\text{scalar}}{\Phi(y)}$$

$$\int d^4x \sqrt{g} \, \underbrace{\delta(x-y)/\sqrt{g}}_{\text{scalar.}} \, \Phi(x) = \Phi(y)$$

$$\boxed{GR} \qquad J_G^\alpha \equiv \sum e_n \int \frac{1}{\sqrt{g(x)}} \delta^4(x^\mu - x_n^\mu) \frac{\partial x^\alpha}{\partial \tau} d\tau$$

What does this mean in terms of charge conservation?

$$D_\alpha J^\alpha \overset{?}{=} 0$$

This can be rewritten as:

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\,V^\mu\right) = 0$$

So the proof from SR carries over and we find

$$D_\mu j^\mu = 0 \qquad \boxed{GR}$$

It also allows to construct a conserved charge:

$$0 = \int d^3x \sqrt{g}\, D_\mu j^\mu = \int d^3x\, \partial_\mu\left(\sqrt{g}\,V^\mu\right)$$

$$= \int d^3x \left(\partial_0 \sqrt{g}\, V^0\right) = \partial_0 \underbrace{\int d^3x \sqrt{g}\, V^0}_{Q}$$

Notice that this does not work for tensors,
i.e. the fact that

$$\nabla_\mu T^{\mu\nu} = 0$$

does not imply the existence of four
conserved charges in GR
(unless T is antisymmetric, see exercises).

Thermodynamics (in SR)

In thermodynamics the number of particles is very large and instead of a sum over individual particles, the system is typically described by particle distribution functions

$$f_{FD} = \frac{1}{e^{E/T} + 1} \qquad (\hbar = 1, k = 1)$$

$$f_{BE} = \frac{1}{e^{E/T} - 1}$$

This construction implies a rest frame of the plasma and one can generalize these expressions by introducing the 'plasma four vector'

$$u_\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ in the rest frame of the plasma

$$u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = u_\nu \eta^{\mu\nu}$$

$$\hookrightarrow E/T \rightarrow -p^\mu u_\mu / T$$

$$f(p^\mu, u^\mu) = \ldots$$

So if p and u are transformed properly,
the particle distribution function is a scalar.

What is the total four-momentum of the plasma?

$$J^\mu \propto \int d^4p \; p^\mu \; f(-p^\mu u_\mu) \; \delta(p^2 - m^2)$$

$$\propto \underbrace{\int \frac{d^3p}{2E}}_{\text{scalar integral measure}} \; p^\mu \, f(-p^\mu u_\mu) \Big|_{p^0 = E = \sqrt{\vec{p}^2 + m^2}}$$

$$J^\mu(u^\mu, T) \propto + u^\mu n \qquad ; \quad u^\mu u_\mu = -1$$

$$n = \int \frac{d^3p}{2E} (-u_\mu p^\mu) f(-p^\mu u_\mu)$$

This can be evaluated in the restframe of the plasma:

$$n = \int \frac{d^3p}{E} E \, f(E, T)$$

The energy momentum tensor is the same construction.

$$T^{\mu\nu} \propto \underbrace{\int \frac{d^3\vec{p}}{E}}_{} p^\mu p^\nu \, f(-p^\mu u_\mu) \Big/$$

$$\propto \int d^4p \, \delta(p^2 - m^2)$$

$p^0 = E = \sqrt{\vec{p}^2 + m^2}$

pressure

$$T^{\mu\nu}(u^\mu, T) = u^\mu u^\nu \omega + \eta^{\mu\nu} \underline{P}$$

$$u_\mu u_\nu T^{\mu\nu} = \omega - P = \int \frac{d^3\vec{p}}{E} p^\mu p^\nu u_\mu u_\nu \, f \Big/$$

$$\overset{R.F.}{=} \int d^3p \, E \cdot f = \rho$$

energy density

$$\eta_{\mu\nu} T^{\mu\nu} = -\omega + 4\underline{P}$$

$$= m^2 \int \frac{d^3p}{E} f$$

$\omega = \underline{P} + \rho$ = enthalpy

In an extensive system (where all thermodynamic potentials are proportional to volume)

$$s = T\frac{\partial p}{\partial T} - p \qquad \text{entropy density}$$

$$\omega = T\frac{\partial p}{\partial T} = T \cdot s$$

So now imagine, that the plasma is in local equilibrium, so it has the form above, but the four-velocity and the temperature can change:

$$T^{\mu\nu}(x^\mu) = \omega(T(x))\, u^\mu u^\nu(x) - p(T(x))\, \eta^{\mu\nu}$$

The fluid equations (c.f. Navier-Stokes and continuity) are then given by

$$\partial_\mu T^{\mu\nu}(x) = 0$$

$$u^\mu = \gamma\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$$

In GR:

$$T^{\mu\nu}(x^\mu) = \omega(T)\, u^\mu u^\nu - p(T)\, g^{\mu\nu}$$

$$\nabla_\mu T^{\mu\nu} = 0$$