A) Linear combination of a weight w tensor density and a weight w tensor density -> weight w tensor density

B) Direct product of a weight w1 tensor density with a weight w2 tensor density gives a tensor density with weight w1+w2

C) Contraction does not change the weight of a tensor density. Likewise lowering/raising indices does not change the weight.

transformation properties of an affine connection

Remember the Christoffel symbol:

$$\left\{ {\kappa \atop \mu\nu} \right\} \equiv \frac{1}{2} g^{\lambda\kappa}\left[\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} + \frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \right]$$

How does this transform:

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$$

$$\left\{ {\kappa \atop \mu\nu} \right\}' = \frac{1}{2} g'^{\lambda\kappa}\left[\frac{\partial g'_{\lambda\nu}}{\partial x'^{\mu}} + \frac{\partial g'_{\lambda\mu}}{\partial x'^{\nu}} - \frac{\partial g'_{\mu\nu}}{\partial x'^{\lambda}} \right]$$

$$= \frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} \frac{\partial x'^{\kappa}}{\partial x^{\alpha}} \left\{ {\alpha \atop \beta\gamma} \right\} + \frac{\partial x'^{\kappa}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}}$$

So the Christoffel symbol is not a tensor, and it transforms non-homogeneously.

Actually, any symbol that transforms as above is called an <u>affine connection</u>.

The Christoffel symbol is one special affine connection, but they are not unique. Adding a tensor gives another aff. connection

$$\Gamma^{\mu}{}_{\nu\lambda} = \left\{ {\mu \atop \nu\lambda} \right\} + T^{\mu}{}_{\nu\lambda}$$

This affine connection might not even be symmetric in $\lambda \leftrightarrow \nu$

This would be called torsion.

In the following we assume that the affine connection is symmetric (torsion=0).

One important relation concerning the second derivative of the coordinate transformation is:

$$\frac{dx^{\alpha}}{dx'^{\beta}} \frac{dx'^{\beta}}{dx^{\beta}} = \delta^{\alpha}{}_{\beta}$$

$$0 = \frac{\partial}{\partial x^{\kappa}} \left(\quad \right) = \frac{\partial x^{\alpha}}{\partial x'^{\beta} \partial x'^{\sigma}} \frac{\partial x'^{\sigma}}{\partial x^{\kappa}} \frac{\partial x'^{\beta}}{\partial x^{\beta}} + \frac{\partial x^{\alpha}}{\partial x'^{\beta}} \frac{\partial^2 x'^{\beta}}{\partial x^{\beta} \partial x^{\kappa}}$$

[Remember: $\frac{d}{d\lambda}\left(M(\lambda)^{-1}\right) = -M^{-1}\frac{d}{d\lambda}M(\lambda)\,M^{-1}$]

$$M M^{-1} = \mathbb{1}$$

$$\frac{d}{d\lambda}\left(M M^{-1}\right) = 0$$

Free falling particle:

We have seen before, that the particle in a gravitational field follows $x^\mu(\tau)$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

with $\quad d\tau^2 = -g_{\mu\nu}\,dx^\mu dx^\nu$

This construction works because when a coordinate transformation is performed, the first term produced an inhomogeneous contribution that is compensated by the affine connection.

This works for any affine connection! Moreover, in the free falling frame, we would get: $(\Gamma = 0)$

$$\frac{d^2 x^\mu}{d\tau^2} = 0 \quad ; \quad d\tau^2 = -\eta_{\mu\nu}\,dx^\mu dx^\nu$$

Hence, the geodesic equation describes the particle in any frame, including gravity.

Covariant derivative:

Consider a contravariant vector:

$$V'^{\mu} = \frac{dx'^{\mu}}{dx^{\rho}} V^{\rho}$$

How does the derivative of the vector transform?

$$\begin{aligned}
\left(\frac{\partial}{\partial x^{\kappa}} V^{\mu}\right)' &= \frac{\partial}{\partial x'^{\kappa}} \left(\frac{\partial x'^{\mu}}{\partial x^{\rho}} V^{\rho}\right) \\
&= \frac{\partial x^{\alpha}}{\partial x'^{\kappa}} \frac{\partial}{\partial x^{\alpha}} \left(\qquad \right) \\
&= \frac{\partial x^{\alpha}}{\partial x'^{\kappa}} \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\alpha}} V^{\rho} \\
&\quad + \frac{\partial x^{\alpha}}{\partial x'^{\kappa}} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial V^{\rho}}{\partial x^{\alpha}}
\end{aligned}$$

This means that $\partial_{\kappa} V^{\mu}$ is not a tensor.

But we can construct one, using the affine connection:

$$\left(\frac{\partial}{\partial x^{\kappa}} V^{\mu} + \Gamma^{\mu}{}_{\rho\kappa} V^{\rho}\right) \equiv T^{\mu}{}_{\kappa}$$

This is a tensor, because the inhomogenous terms cancel. (see exercise)

$$V^{\mu}{}_{;k} \equiv \nabla_k V^{\mu} \equiv \left(\partial_k V^{\mu} + \Gamma^{\mu}{}_{k\lambda} V^{\lambda}\right)$$

$$\frac{\partial}{\partial x^{\lambda}} V \equiv \partial_{\lambda} V = V_{,\lambda}$$

Notice that $\nabla_{\lambda} V^{k}$ is by no means proportional to V^{k}

Covariant derivative of a covariant vector

$$V_\mu{}' = \frac{\partial x^\alpha}{\partial x'^\mu} V_\alpha{}'$$

$$(\partial_\nu V_\mu)' = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \partial_\beta V_\alpha$$

$$+ \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} V_\alpha$$

Using the transformation properties of the affine connection again, we can construct a covariant derivative

$$\nabla_\nu V_\mu \equiv \partial_\nu V_\mu - \Gamma^\alpha_{\mu\nu} V_\alpha$$

covariant derivative of tensors

$$\nabla_\kappa T^\alpha{}_\beta \equiv \partial_\kappa T^\alpha{}_\beta + \Gamma^\alpha{}_{\kappa\mu} T^\mu{}_\beta$$

$$- \Gamma^\mu{}_{\kappa\beta} T^\alpha{}_\mu$$

$$\nabla_\kappa X^{\alpha\beta\gamma}{}_{\delta\epsilon} = \partial_\kappa X + 5 \times \Gamma \times X \times (\pm)$$

Notice that

$$
\begin{aligned}
\nabla_\kappa \delta^\alpha{}_\beta &= \cancel{\partial_\kappa \delta^\alpha{}_\beta} \\
&\quad + \Gamma^\alpha{}_{\mu\kappa}\, \delta^\mu{}_\beta \\
&\quad - \Gamma^\mu{}_{\kappa\beta}\, \delta^\alpha{}_\mu \\
&= \Gamma^\alpha{}_{\beta\kappa} - \Gamma^\alpha{}_{\kappa\beta} = 0
\end{aligned}
$$

Also

$$
\partial_\mu g_{\alpha\beta} = \Gamma^\lambda{}_{\mu\alpha}\, g_{\lambda\beta} + \Gamma^\lambda{}_{\mu\beta}\, g_{\alpha\lambda}
$$

$$
\rightarrow \quad \nabla_\mu g_{\alpha\beta} = 0
$$

But notice that this is based on the fact that we used the Christoffel symbols for the affine connection.

In general, the covariant derivative also fulfills the Leibniz rule:

$$
\nabla_\kappa (A^\lambda B_\mu) = (\nabla_\kappa A^\lambda)\, B_\mu + A^\lambda (\nabla_\kappa B_\mu)
$$

Together, this implies

$$T^{\mu\nu\lambda} = g^{\mu\kappa} T_{\kappa}{}^{\nu\lambda}$$

$$D_\rho T^{\mu\nu\lambda} = \cancel{(D_\rho g^{\mu\kappa})} T_{\kappa}{}^{\nu\lambda} + g^{\mu\kappa} (D_\rho T_{\kappa}{}^{\nu\lambda})$$

The same holds true for contractions:

$$T^{\mu} = T^{\mu\nu\alpha}{}_{\alpha} = T^{\mu\nu\alpha\beta} g_{\alpha\beta}$$

$$\rightarrow \quad D_\rho T^{\mu\nu} = g_{\alpha\beta} D_\rho T^{\mu\nu\alpha\beta}$$

In total, this gives one a easy receipt
for the principle of convariance:

Start from a theory without gravity and
do the replacement

$$\boxed{\begin{aligned} \eta &\rightarrow g \\ \partial_\mu &\rightarrow D_\mu \end{aligned}}$$