Notice that this metric can also be brought to the more familiar form from cosmology by using the coordinate transformation (K>0)

$$t = \frac{1}{\sqrt{K}}\left[\frac{Kx}{2}\cosh(\sqrt{K}t') + \left(1+\frac{Kx^2}{2}\right)\sinh(\sqrt{K}t)\right]$$

$$x^2 = \vec{x}\cdot\vec{x} \qquad ; \eta_{\mu\nu} = \begin{pmatrix} 1 & \\ & -\mathbb{1} \end{pmatrix}$$

$$x^\mu = x'^\mu \cdot \exp(\sqrt{K}t)$$

In the new coordinates the line elements reads

$$ds^2 = dt'^2 - a^2(t')\, dx' \cdot dx'$$

with

$$a(t) = \exp \sqrt{K}t$$

Reminder: dS space in cosmology

Remember that we found this metric earlier as a solution to the Einstein equation with a cosmological constant.

Lagrangian/action

$$S = M_{Pl}^2 \int d^4x \sqrt{g} \left\{ R + \lambda \right\}$$

$$\hookrightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda\, g_{\mu\nu} = T_{\mu\nu} \times 8\pi G_N$$

In vacuum (CC only)

$$R = 4\lambda = N(N-1)\, K = 12K$$

$$\rightarrow \quad \lambda = 3K$$

$\lambda > 0$ de Sitter

$\lambda < 0$ anti-de Sitter

$\lambda = 0$ Minkowski

remember that in general an isotropic metric can be grought to the FRW form

$$ds^2 = -dt^2 + a(t)^2 \, d\vec{x} \cdot d\vec{x}$$

and the Einstein equation becomes the Friedman equation

$$H^2 = \frac{8\pi}{3} G_N \rho \qquad H = \dot{a}/a$$

$$\hookrightarrow H^2 = \frac{\lambda}{3} = K$$

# Properties of deSitter space

Besides the one we one we just constructed, there are many different coordinate systems to represent deSitter space and they all have different advantages and disadvantages.

## Parameterization of the unit sphere

Consider the parameterization of the unit sphere, as e.g. the Euler angles in 3D

$$\omega^i : \sum_i \omega^{i\,2} = 1$$

$$\omega_1 = \cos\theta_1$$
$$\omega_2 = \sin\theta_1 \cos\theta_2$$
$$\vdots$$
$$\omega_{d-1} = \sin\theta_1 \ldots \sin\theta_{d-2}\cos\theta_{d-1}$$
$$\omega_d = \sin\theta_1 \ldots \sin\theta_{d-2}\sin\theta_{d-1}$$

with

$$0 \leq \theta_i \leq \pi \qquad i \in 1..(d-1)$$

$$0 \leq \theta_{d-1} \leq 2\pi$$

The line element then reads

$$d\Omega_d^2 = \sum_i d\omega_i^2 = d\theta_1^2 + \sin^2\theta_1 \, d\theta_2^2$$
$$+ \ldots + \sin^2\theta_1 \ldots \sin^2\theta_{d-1} \, d\theta_d^2$$

This can be then used to provide another parameterization of deSitter space.

A] global coordinates:

$$X^M : \quad X^0 = r \cdot \sinh \tau/r$$
$$X^i = r \cdot \omega^i \cdot \cosh \tau/r$$

$$\hookrightarrow \quad X^M X^N \eta_{MN}$$
$$= -X^{0\,2} + \sum_i X^{i\,2} = r^2$$

So the parameters

$$\tau, \omega_i \quad (i \in 1..(d-1))$$

provide the d coordinates that parameterize deSitter.

$$ds^2 = -dX^{0\,2} + dX_i^2$$
$$= -d\tau^2 + r^2 \cosh^2 \tau/r \; d\Omega_{d-1}^2$$

Notice here that the angles parameterize a sphere that fulfills

$$\sum_i \omega_i^2 = \text{const} \Rightarrow \sum_i \omega_i d\omega_i = 0$$

In global coordinates, deSitter space looks like a sphere that goes from infinite size at $\tau = -\infty$ to finite size at $\tau = 0$ and back to infinite size at $\tau \approx \infty$ .

B] conformal coordinates

$$\cosh \tau/r = \frac{1}{\cos T/r}$$

$$\cosh > 0 \Rightarrow T/e \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\hookrightarrow \sinh \tau/r \frac{d\tau}{r} = \frac{-1}{\cos^2 \frac{T}{r}} \sin \frac{T}{r} \frac{dT}{r}$$

Notice that

$$\sinh \frac{\tau}{r} = \sqrt{\cosh^2 \frac{\tau}{r} - 1} = \sqrt{\frac{1}{\cos^2} - 1} = \tan T/e$$

$$\hookrightarrow \frac{dT}{r} = \frac{1}{\cos T/r} \frac{d\mathcal{T}}{r}$$

$$d\tau^2 = \frac{1}{\cos^2} d\mathcal{T}^2$$

$$\hookrightarrow ds^2 = \frac{1}{\cos^2 T/r}\left[-d\mathcal{T}^2 + \ell^2 d\Omega^2\right]$$

Notice that this metric is related to the usual flat (compact) space by a Weyl transformation (which is no coord. trafo.)

$$\bar{g} = \omega g$$

Notice again that if two metrics are related by a conformal/Weyl transformation that they have the same null geodesics = paths of light

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

But also in general, consider the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\kappa} \left\{ \frac{\partial g_{\alpha\kappa}}{\partial x^\beta} + \frac{\partial g_{\beta\kappa}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\kappa} \right\}$$

$$\bar{g} = \omega g$$

$$\hookrightarrow \bar{\Gamma}^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

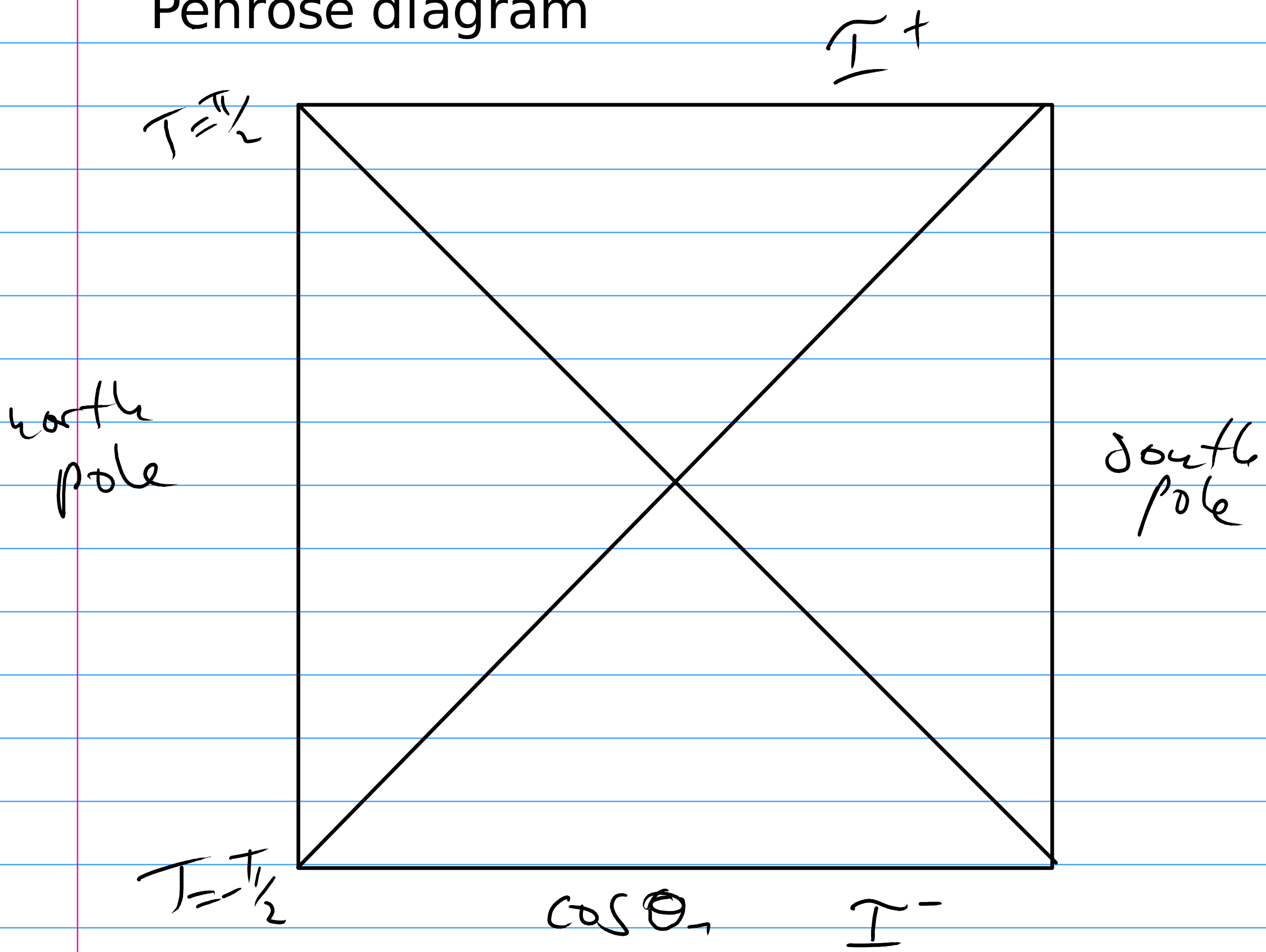
$$+ \omega^{-1} \frac{\partial x^\mu}{\partial \tau} \frac{dx^\kappa}{\partial \tau} \frac{\partial \omega}{\partial x^\kappa}$$

$$\underbrace{- \frac{1}{2} g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tau} \frac{\partial x^\beta}{\partial \tau}}_{=0} \cdot g^{\mu\kappa} \frac{\partial \omega}{\partial x^\kappa}$$

$$= \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \frac{dx^\mu}{d\tau} \partial_\tau \log \omega(t)$$

The last term can be absorbed into a reparameterization of the path parameter $\tau$

Penrose diagram




C] planar coordinates again:

$$X^0 = \sinh t - \frac{1}{2} x_i x^i e^{-t}$$

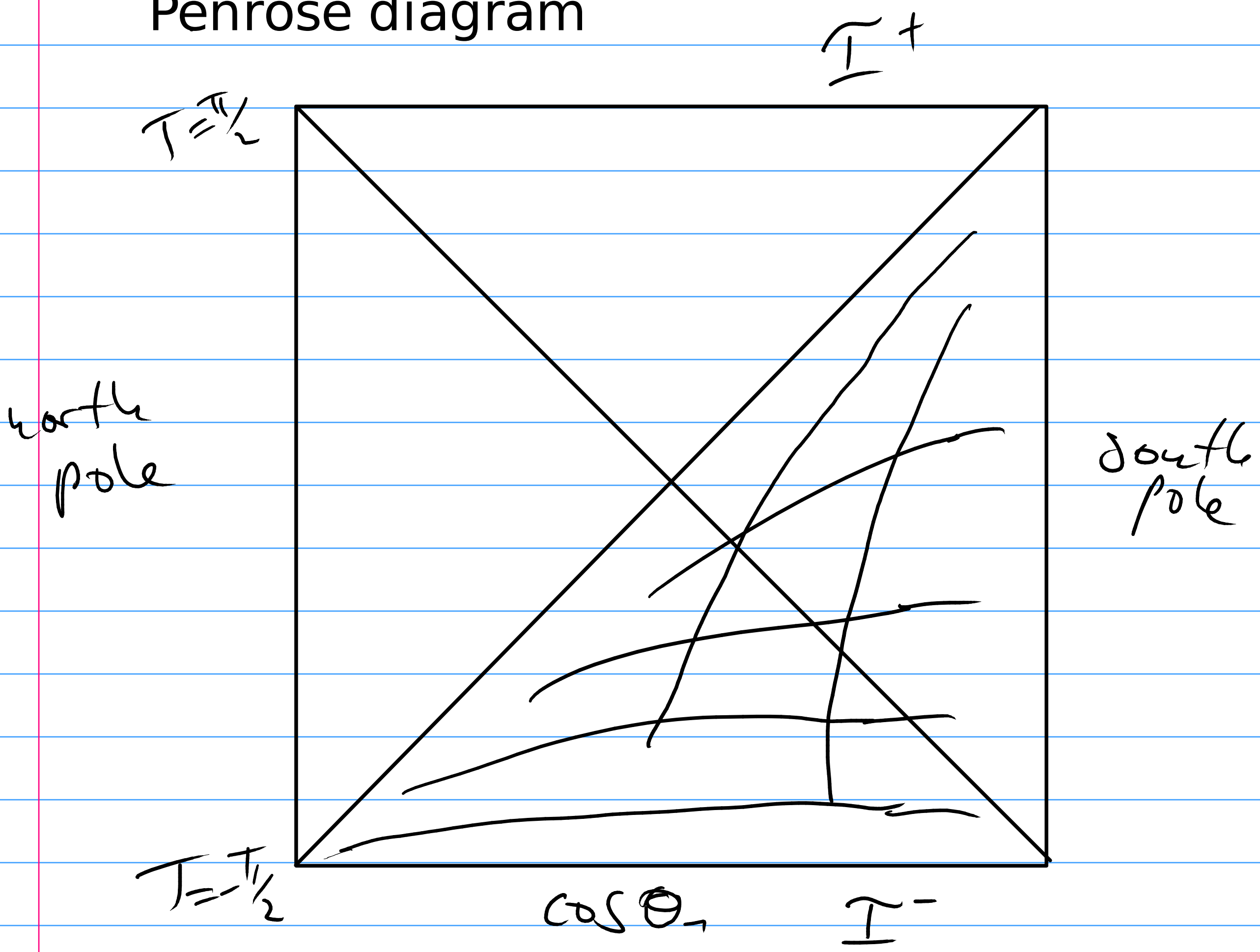
$$X^i = x^i e^{-t}$$

$$X^d = \cosh t - \frac{1}{2} x_i x^i e^{-t}$$

$$ds^2 = -dt^2 + e^{-2t} dx_i dx^i$$

This parameterization actually only covers half of the original parameterization

Penrose diagram




There are horizons in this space.
The parameterization of the other half was discussed before.