Back to the Lorentz group.

$$\vec{a}^2 = A(A+1)$$
$$\vec{b}^2 = B(B+1)$$

A, B are positive and either integer or half-integer.

This means that all representations of the Lorentz group can be characterized by (A,B).

For example, for a vector $(A,B) = \left(\frac{1}{2}, \frac{1}{2}\right)$

For a tensor, there are components that transform as

$(1,1)$, $(1,0)$, $(0,1)$, $(0,0)$

Weyl spinors are

$\left(0, \frac{1}{2}\right)$, $\left(\frac{1}{2}, 0\right)$ } right/left-handed 2 component

Dirac spinor:

$$\left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right)$$

4 component

Consider the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\cdot \mathbb{1}\cdot \eta^{\mu\nu}$$

For example

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}$$

σ^i Pauli matrices

Then $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$

fulfill the Poincare algebra

$$U = \exp\left[\frac{i}{8}[\gamma_\mu, \gamma_\nu]\,\omega^{\mu\nu}\right]$$

$\psi \to U\psi$ is a representation

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \begin{matrix} \} \text{ left} & (\tfrac{1}{2}, 0) \\ \} \text{ right} & (0, \tfrac{1}{2}) \end{matrix}$$

Spinors are not true representations of the Lorentzgroup because rotation around a fixed axis by 360^0 gives -1.

$$\therefore \bar{\psi}\psi = \psi^{+}\gamma^{0}\psi \quad \text{is a scalar}$$

The γ^0 is required because the spinor representation is not unitary.

$$U^{+} = \gamma_0 U^{-1} \gamma_0$$

Spinors in GR

For vectors and tensors, the general coordinate transformations reduce to the Lorentz transformations

When $\frac{\partial x'}{\partial x} = \text{const} \rightarrow \Lambda^{\alpha}{}_{\beta}$

For the spinors, one would need
a represenation of the general coordinate transformations GL(4) that reduce
to the spinor represention if the coordinate transformation was linear.

Unfortunately, there no representations of GL(4) with this property.

So in order to implement spinors, one has to formulate GR in a different way, using vielbeins. Given a metric g, one can decompose the metric as

$$g_{\mu\nu} = V_\mu{}^\alpha V_\nu{}^\beta \eta_{\alpha\beta}$$

Notice that the vielbein is in general not a coordinate transformation, so one cannot always write

$$V_\mu{}^\alpha(x) \not= \frac{\partial x^\alpha}{\partial x'^\mu}$$

E.g. $\partial_\nu V_\mu{}^\alpha$ does not need to be symmetric

The vielbein transforms as

$$V'_\mu{}^\alpha \to \frac{dx^\nu}{dx'^\mu} V_\nu{}^\alpha$$

So the vielbein should be understood as a collection of four covariant vectors.

So instead of using the metric as a dynamical degree of freedom, we can write everyting in terms of vielbeins.

In principle, we can now expand all the tensors in terms of the vielbeins:

$$A_\mu = V_\mu^{\ \alpha} \bar{A}_\alpha$$

Then $\bar{A}_\alpha$ does not change under coordinate transformations!

Also, e.g.

$$A_\mu B_\nu g = \bar{A}_\alpha \bar{B}_\beta \underbrace{V_\mu^{\ \alpha} V_\nu^{\ \beta} g^{\mu\nu}}_{\eta^{\alpha\beta}}$$

$$= \bar{A}_\alpha \bar{B}_\beta \eta^{\alpha\beta}$$

$$\mathcal{L}(g, A^\mu) = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$\not{\downarrow}$ covariant derivatives

$$\mathcal{L}(\eta, \bar{A}) = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Notice that the theory cannot remove all reference to the metric due to the covariant derivatives ->
construct the covariant derivatives in terms of the vielbeins.

Also notice that if the theory is written down in quantities with a bar, there is an additional Lorentz invariance!

$$A_\mu B_\nu g^{\mu\nu} = \bar{A}_\alpha \bar{B}_\beta \eta^{\alpha\beta}$$

This e.g. is invariant under

$$\left.\begin{array}{l} \bar{A} \to \Lambda \bar{A} \\ \bar{B} \to \Lambda \bar{B} \end{array}\right\} \quad \bar{A}^\alpha \bar{B}^\beta \eta_{\alpha\beta} \to \bar{A}^\alpha \bar{B}^\beta \eta_{\alpha\beta}$$

All quantities have now different indices.
There are μ, ν, λ .. $GL(4)$

α, β, γ .. $SO(1,3)$

$V_\mu{}^\alpha$ (α ← $SO(1,3)$, μ ← $GL(4)$)

$$V_{\mu\alpha} = V_\mu{}^\beta \eta_{\beta\alpha}$$

$$V^{\mu\alpha} = g^{\mu\nu} V_\nu{}^\alpha$$

$$V^{\mu}{}_{\alpha} V_{\mu}{}^{\beta} = \delta_{\alpha}{}^{\beta}$$

$$V_{\alpha}{}^{\mu} V^{\alpha}{}_{\nu} = \delta^{\mu}{}_{\nu}$$

And indices can be transformed into each other with the vielbeins

$$A_{\mu} V^{\mu}_{\alpha} = \bar{A}_{\alpha}$$

Covariant derivative

Consider the covariant derivative of a vector.

$$D_{\mu} A^{k} = \partial_{\mu} A^{k} + \Gamma^{k}{}_{\mu\nu} A^{\nu}$$

Obviously the Christoffel symbol can be written in terms of the Vielbein.

The covariant derivative is constructed to transform as a tensor

$$D_{\mu} A^{k} \to \frac{\partial x'^{k}}{\partial x^{\lambda}} \frac{\partial x^{\nu}}{\partial x'^{\mu}} D_{\nu} A^{\lambda}$$

But now we can also define a covariant derivative with respect to SO(1,3)

$$D_\mu A^\lambda = D_\mu V^\lambda_{\ \alpha} \bar{A}^\alpha$$

$$\equiv V^\lambda_{\ \alpha} V^\beta_{\ \mu} D_\beta \bar{A}^\alpha$$

So

$$D_\beta \bar{A}^\gamma = V^\gamma_{\ \lambda} V_\beta^{\ \mu} D_\mu V^\lambda_{\ \alpha} \bar{A}^\alpha$$

$$= V^\gamma_{\ \lambda} V^\mu_{\ \beta} \Big((\partial_\mu V^\kappa_{\ \alpha}) \bar{A}^\alpha + V^\kappa_{\ \alpha} \partial_\mu \bar{A}^\alpha + \Gamma^\kappa_{\ \mu\lambda} V^\lambda_{\ \alpha} \bar{A}^\alpha \Big)$$

By construction this transforms as

$$D_\beta \bar{A}^\gamma \to \Lambda^\alpha_{\ \beta} \Lambda^\gamma_{\ \delta} D_\alpha \bar{A}^\delta$$

This is equivalent and will lead to the same EoM for vectors/tensors but will generalize to representations of SO(1,3) that are not embedded into GL(4).