General representations of the Lorentz transformations

Consider a field ψ with some indices n (these are not necessarily the usual indices from vectors/tensors).

If the field ψ is a representation of the Lorentz group then

$$\psi'_m = [D(\Lambda)]_{mn}\,\psi_n$$

One consequence is that

$$[D(\Lambda_1\Lambda_2)]_{mn} = [D(\Lambda_1)]_{me}\,[D(\Lambda_2)]_{en}$$

For example:

vector: $[D(\Lambda)]_{mn} = \Lambda^{\alpha}{}_{\beta}$ $\quad m=\alpha \quad n=\beta$

tensor $[D(\Lambda)]_{mn} = \Lambda^{\alpha}{}_{\beta}\Lambda^{\gamma}{}_{\delta}$ $\quad m=(\alpha,\gamma)$ $\quad n=(\beta,\delta)$

$$T \to DT = D^{\alpha\gamma}_{\beta\delta}\,T_{\alpha\gamma}$$

In fact, the tensor representations are the only true representations of the Lorentz group.

Instead, one can look for the representations of infinitesimal group elements:

$$\Lambda^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta} \qquad |\omega| \ll 1$$

Then

$$\Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} \eta_{\alpha\beta} = \eta_{\gamma\delta}$$

$$\Rightarrow \boxed{\omega_{\gamma\delta} = - \omega_{\delta\gamma}}$$

Then any representation can be written in the form

$$[D(\Lambda)]_{mn} = 1 + \frac{1}{2} \omega^{\alpha\beta} [\sigma_{\alpha\beta}]_{mn}$$

σ are then the generators of the representation.

The multiplication rule then implies

$$D(\Lambda)\, D(1+\omega)\, D(\Lambda^{-1}) = D(1 + \Lambda \omega \Lambda^{-1})$$

$$\Rightarrow \quad D(\Lambda)\, \sigma_{\alpha\beta}\, D(\Lambda^{-1}) = \sigma_{\gamma\delta} \Lambda^{\gamma}{}_{\alpha} \Lambda^{\delta}{}_{\beta}$$

If we now also consider Λ is close to unity.

$$\hookrightarrow [\sigma_{\alpha\beta}, \sigma_{\gamma\delta}] = \eta_{\gamma\beta}\sigma_{\alpha\delta} + \eta_{\gamma\alpha}\sigma_{\delta\beta} + \eta_{\delta\beta}\sigma_{\gamma\alpha} + \eta_{\delta\alpha}\sigma_{\beta\gamma}$$

This is called the Dirac algebra.

The Dirac algebra can be written in a more familiar form:

$$a_i = \frac{1}{2}\left[-i\,\epsilon_{ijk}\,\sigma_{jk} + \sigma_{i0}\right]$$

$$b_i = \frac{1}{2}\left[\quad - \quad\right]$$

The algebra in terms of a and b:

$$[a_i, a_j] = i\,\epsilon_{ijk}\,a_k$$

$$[b_i, b_j] = i\,\epsilon_{ijk}\,b_k$$

$$[a, b] = 0$$

So a and b a generators of SO(3).

Hence:

$$\vec{a}^2 = A(A+1)$$
$$\vec{b}^2 = B(B+1)$$

A, B are positive and either integer or half-integer.

This means that all representations of the Lorentz group can be characterized by (A,B).

For example, for a vector $(A,B) = \left(\frac{1}{2}, \frac{1}{2}\right)$

For a tensor, there are components that transform as

$$(1,1), \quad (1,0), \quad (0,1), \quad (0,0)$$

Weyl spinors are

$$\left.\begin{array}{l} \left(0, \frac{1}{2}\right) \\ \left(\frac{1}{2}, 0\right) \end{array}\right\} \begin{array}{c} \text{right/left-handed} \\ \text{2 component} \end{array}$$

Dirac spinor: $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$

4 component

$$\psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \begin{matrix} \} \text{ left} & (\tfrac{1}{2}, 0) \\ \} \text{ right} & (0, \tfrac{1}{2}) \end{matrix}$$

Spinors are no true representations of the Lorentzgroup because rotation around a fixed axis by 360° gives -1.

$\psi^N(x^\mu)$: $\bar{\psi}\psi = \psi^\dagger \gamma^0 \psi$ is a scalar.

The γ^0 is required because the spinor representation is not unitary.

Spinors in GR

For vectors and tensors, the general coordinate transformations reduce to the Lorentz transformations

When $\frac{\partial x'}{\partial x} = \text{const} \rightarrow \Lambda^\alpha{}_\beta$

For the spinors, one would need
a represenation of the general coordinate transformations GL(4) that reduce
to the spinor represention if the
coordinate transformation was linear.

Unfortunately, there no representations of GL(4) with this property.

So in order to implement spinors, one has to formulate GR in a different way, using vielbeins. Given a metric g, one can decompose the metric as

$$g_{\mu\nu} = V_\mu{}^\alpha V_\nu{}^\beta \eta_{\alpha\beta}$$

Notice that the vielbein is in general not a coordinate transformation, so one cannot always write

$$V_\mu{}^\alpha(x) \neq \frac{\partial x^\alpha}{\partial x'^\mu}$$

E.g. $\partial_\nu V_\mu{}^\alpha$ does not need to be symmetric

The vielbein transforms as

$$V'_\mu{}^\alpha \to \frac{dx^\nu}{dx'^\mu} V_\nu{}^\alpha$$

So the vielbein should be understood as a collection of four covariant vectors.

So instead of using the metric as a dynamical degree of freedom, we can write everyting in terms of vielbeins.

In principle, we can now expand all the tensors in terms of the vielbeins:

$$A_\mu = V_\mu{}^\alpha \bar{A}_\alpha$$

Then $\bar{A}_\alpha$ does not change under coordinate transformations!

Also, e.g.

$$A_\mu B_\nu g = \bar{A}_\alpha \bar{B}_\beta \underbrace{V_\mu^\alpha V_\nu^\beta g^{\mu\nu}}_{\eta^{\alpha\beta}}$$

$$= \bar{A}_\alpha \bar{B}_\beta \eta^{\alpha\beta}$$

$$\mathcal{L}(g, A^\mu) = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$\not\downarrow$ covariant derivatives

$$\mathcal{L}(\eta, \bar{A}) = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Notice that the theory cannot remove all reference to the metric due to the covariant derivatives ->
construct the covariant derivatives in terms of the vielbeins.

Also notice that if the theory is written down in quantities with a bar, there is an additional Lorentz invariance!

$$A_\mu B_\nu g^{\mu\nu} = \bar{A}_\alpha \bar{B}_\beta \eta^{\alpha\beta}$$

This e.g. is invariant under

$$\left.\begin{aligned} \bar{A} &\to \Lambda \bar{A} \\ \bar{B} &\to \Lambda \bar{B} \end{aligned}\right\} \quad \bar{A}\bar{B}\eta \to \bar{A}\bar{B}\eta$$

All quantities have now different indices.
There are μ, ν, λ .. $GL(4)$

α, β, γ .. $SO(1,3)$

$V_\mu{}^\alpha$ $\leftarrow SO(1,3)$

$\nwarrow GL(4)$

$$V_{\mu\alpha} = V_\mu{}^\beta \eta_{\beta\alpha}$$

$$V^{\mu\alpha} = g^{\mu\nu} V_\nu{}^\alpha$$

$$V^{\mu}{}_{\alpha} V_{\mu}{}^{\beta} = \delta_{\alpha}{}^{\beta}$$

$$V_{\alpha}{}^{\mu} V^{\alpha}{}_{\nu} = \delta^{\mu}{}_{\nu}$$

And indices can be transformed into each other with the vielbeins

$$A_{\mu} V^{\mu}{}_{\alpha} = \bar{A}_{\alpha}$$