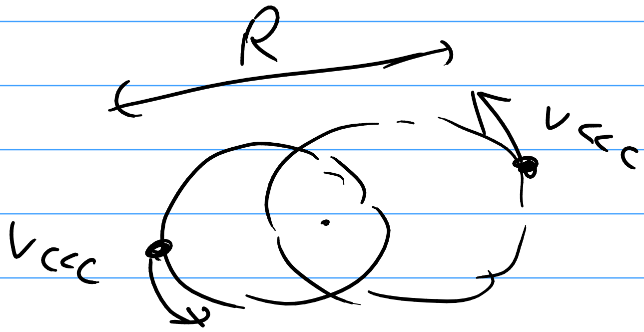
Quadrupole approximation

So far we only assumed weak fields and the wave zone approximation, which is essentially always true.

Now we restrict to localized sources that move relatively slowly, e.g. a binary systems long before merger.



$$\omega R \simeq v \ll 1$$

$$\omega \sim \omega_{planets}$$

So the emitted frequency is small in units of the size of the system.

If we use this fact in the Fourier transform, then in leading order, we can neglect the phases:

$$\tilde{T}_{ij}(\vec{k},\omega) = \int d^3x \; e^{+i\vec{k}\vec{x}} \, \tilde{T}_{ij}(\vec{x},\omega)$$

$$\simeq \int d^3x \; \tilde{T}_{ij}(\vec{x},\omega)$$

Using energy momentum conservation:

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} T_{ij}(\vec{x},\omega) = -\omega^2 T^{00}$$

we find

$$D_{ij}(\omega) = \int d^3x \; x^i x^j \; T^{00}$$

$$= \frac{-1}{\omega^2}\int d^3x \; x^i x^j \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_m} T_{lm}$$

$$= -\frac{2}{\omega^2}\int d^3x \; T_{ij} \simeq -\frac{2}{\omega^2} T_{ij}(\vec{k},\omega)$$

And the radiated power

$$\frac{dP}{d\Omega} = \frac{G\omega^6}{4\pi} \Lambda_{ij,lm}(\hat{k}) \; D_{ij}^* D_{lm}(\omega)$$

or for a spectrum

$$\frac{dE}{d\Omega} \longrightarrow \frac{1}{2} G \Lambda_{ij,lm}$$

$$\times \int_0^\infty d\omega \, \omega^6 \; D_{ij}^* D_{lm}(\omega)$$

Since ther quadrupole moment does not depend on the momentum, one can perform the integral of the solid angle:

$$\int d\Omega \, \hat{k}_i \hat{k}_j = \frac{4\pi}{3} \delta_{ij}$$

$$\int d\Omega \, \hat{k}_i \hat{k}_j \hat{k}_l \hat{k}_m = \frac{4\pi}{15}$$

$$\times \left(\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl} \right)$$

$$\hookrightarrow \quad P = \frac{2G\omega^6}{5} \left[D_{ij} D_{ij}^* - \frac{1}{3} |D_{ii}|^2 \right]$$

quadrupole moment of a binary system:

$$\bar{X}^\mu(\lambda)$$

$$T^{\mu\nu}(\vec{x},t) = \sum_{\text{particles}} m \int d\lambda \, \frac{\partial x^\mu}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda} \, \delta^4\left(x^\mu - \bar{X}^\mu(\lambda) \right)$$

non-relativistic:

$$\simeq \sum m \frac{\partial x^\mu}{\partial t} \frac{\partial x^\nu}{\partial t} \delta^3(\vec{x} - \vec{X}(t))$$

$$T^{00} \simeq \sum_{\text{particles}} m \, \delta^3(\vec{x} - \vec{X}(t))$$

$$D_{ij}(t) = \int d^3x \; x^i x^j \; T^{00}$$

$$\simeq +2 \begin{pmatrix} c^2 & cs & 0 \\ cs & s^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} R^2 m$$

$c \sim \cos\omega_p t$
$s \sim \sin\omega_p t$

where we used $\quad x^i = R \begin{pmatrix} \cos\omega_p t \\ \sin\omega_p t \\ 0 \end{pmatrix}$

Furthermore

$$\begin{pmatrix} c^2 & cs & 0 \\ cs & s^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cos 2\omega_p t & \sin 2\omega_p t & 0 \\ \sin 2\omega_p t & -\cos 2\omega_p t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \frac{1}{2} \begin{pmatrix} \frac{1}{2} & & \\ & & \\ & & \frac{1}{2} \end{pmatrix}$$

remember:

$$c^2 = \frac{1}{2}(c^2 - s^2 + 1)$$

$$s^2 = \frac{1}{2}(s^2 - c^2 + 1)$$

The constant piece does not produce radiation while the Fourier decomposition of quadrupole moment is

$$\cos 2\omega_p t = \frac{1}{2} e^{2i\omega_p t} + c.c.$$
$$\sin 2\omega_p t = -\frac{i}{2} e^{2i\omega_p t} + c.c.$$

$$\begin{pmatrix} c^2 & cs & 0 \\ cs & s^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{2i\omega_p t} + c.c.$$
$$+ \text{const.}$$

Using this is in the expression of the emitted power

$$P = \frac{2 G\omega^6}{5} \left(D_{ij} D_{ij}^*(\omega) - |D_{ii}|^2 \right)$$

and the Fourier decomposition

$$D_{ij}(\omega = 2\omega_p) = \frac{1}{2} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} mR^2$$

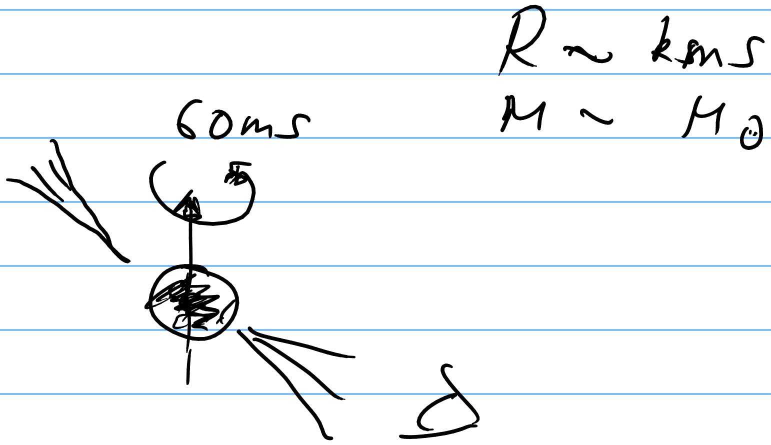
one finally obtains

$$P = \frac{128}{5} G \omega_p^6 R^4 m^2$$

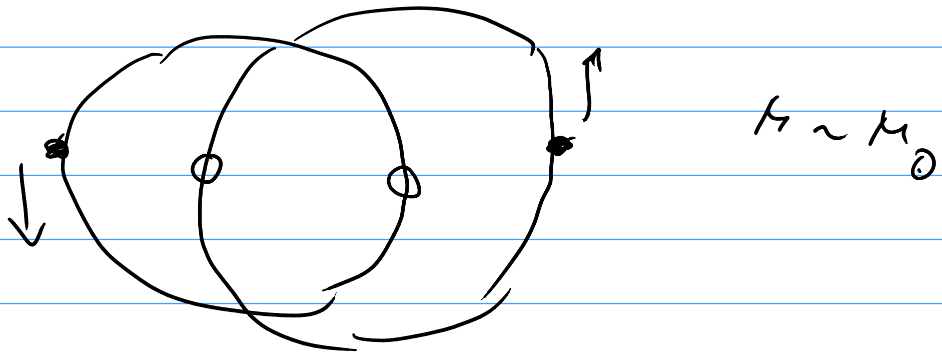
Hulse-Taylor binary

In 1974 Hulse and Taylor discovered a pulsar in a binary system.

A pulsar is a quickly rotating neutron star that emits a radio signal:



An observer might hence see a radio pulse every 60 ms. This provides a very stable signal that acts as a clock.



distance to earth: 21 kly ~ 6.4 kpc
orbital period ~7 hrs
major axis ~ 2 x 10^9 m ~ 10 sec time delay

The pulse signal can be used to measure the orbit of the binary system
(~7 classical parameters)

Observing for a long enough time, one also sees GR effects

- time delatation due to motion
- periastron rotation
- energy loss of the system due to gravitational radiation

figure (Wikipedia):