Generation of gravitational waves

$$h_{\mu\nu}(\vec{x},t) = 4G \int d^3x' \, \frac{S_{\mu\nu}(x', t - |x - x'|)}{|x - x'|}$$

Consider the Fourier transformation of the source

$$S_{\mu\nu}(\vec{x},t) = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{\lambda}{}_{\lambda}$$

$$S_{\mu\nu}(\vec{x},\omega) = \int dt \, e^{i\omega t} S(\vec{x},t)$$

or:

$$S_{\mu\nu}(\vec{x},t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} S(\vec{x},\omega)$$

$$\hookrightarrow \quad h_{\mu\nu} = 4G \int \frac{d\omega}{2\pi} d^3x' \frac{1}{|x-x'|} S_{\mu\nu}(\vec{x}',\omega)$$

$$\times \exp\left\{ -i\omega t + i\omega |x - x'| \right\}$$

Next, we use the wave zone approximation

meaning the observer is quite far away from the source

$$|x'| \ll |x| = r$$

$$|x - x'| = \sqrt{(\vec{x}-\vec{x}')(x-\vec{x}')}$$

$$= \sqrt{x^2 - 2x\cdot x' + x'^2}$$

$$= r - \frac{\vec{x}'\cdot\vec{x}}{r}$$

$$\hookrightarrow h_{\mu\nu} = \int \frac{d\omega}{2\pi} \frac{4G}{r} \exp(i\omega r - i\omega t)$$

$$\times \int d^3x' \, S_{\mu\nu}(x', \omega)\, e^{-i\omega \frac{\vec{x}\cdot\vec{x}'}{r}}$$

Notice that if we identify

$$\vec{k} = \frac{\vec{x}}{r}\cdot\omega \quad ; \quad k^0 = \omega$$

$$h_{\mu\nu}(\vec{x}, t) = \int \frac{d\omega}{2\pi} \, e_{\mu\nu} \, \exp(i k_\mu x^\mu)$$

$$e_{\mu\nu}(k^\mu) = \frac{4G}{r} \int d^3x' \, S_{\mu\nu}(x', \omega)\, e^{-i\vec{k}\cdot\vec{x}'}$$

$$= \frac{4G}{r} S_{\mu\nu}(\vec{k}, \omega) \Big|_{\vec{k} = \omega\frac{\vec{x}}{r}}$$

So in the wave zone approximation,
the weak field limit leads to a plane wave.

Now to simplify, let's consider a single mode
with a frequency ω

$$\int \frac{d\omega}{2\pi} S(\vec{k},\omega) \rightarrow S(\vec{k},\omega)$$

We can then use the wave zone approximation
to calculate the energy flux per solid angle

$$\frac{dP}{d\Omega} = r^2 \left(\frac{x^i}{r}\right) \langle t^{0i} \rangle$$

$$= r^2 \frac{\left(k \cdot \frac{\vec{x}}{r}\right) \omega}{16\pi G}$$

$$\times \left[e^{\lambda\nu *} e_{\lambda\nu} - \frac{1}{2} |e^{\lambda}{}_{\lambda}|^2 \right] (k,\omega)$$

that can be expressed in terms of the
source:

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi} \left[T^{\lambda\nu} T^*_{\lambda\nu} - \frac{1}{2} |T^{\lambda}{}_{\lambda}|^2 \right]$$

Remember that the energy-momentum tensor is conserved, i.e.

$$\partial_\mu T^{\mu\nu} = 0$$

or in Fourier space:

$$k_\mu T^{\mu\nu} = 0 \qquad (k_\mu k^\mu = 0)$$

$\hookrightarrow$ we can remove $T^{0\lambda}$

in favor of T^{ij}

This way, the flux can be written as

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi} \Lambda_{ij,kl} T^{ij*} T^{lk}$$

where Λ is the projection tensor on the transverse traceless part.

$$P_{lm} = \delta_{lm} - \frac{k_l k_m}{\vec{k}^2}$$

and

$$\Lambda_{ij,lm} = P_{il} P_{jm} - \frac{1}{2} P_{ij} P_{lm}$$

P is the projection on the transverse plane

$$P_{lm} k_l = P_{lm} k_m = 0$$

$$P^2 = P \quad ; \quad \mathrm{Tr} P = 2$$

$$\hookrightarrow \Lambda_{ij,lm} k_i = 0 \quad (+ 3 \text{ others})$$

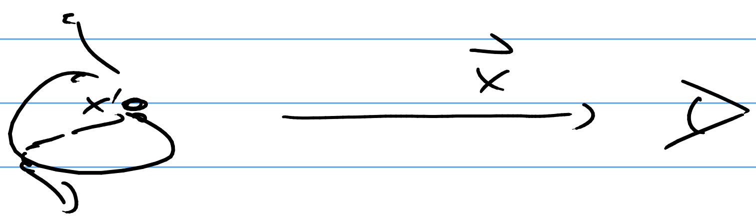
$$\Lambda_{ii,lm} = \Lambda_{ij,mm} = 0$$

This is the power emitted from a single mode with frequency ω

If you would observe the system forever, one can also calculate the total energy radiated for the full spectrum.

$$\frac{dE}{d\Omega} = \frac{G}{\pi} \Lambda_{ij,kl}(\vec{k})$$

$$\times \int \frac{d\omega}{2\pi} \omega^2 \left(T^{ij*} T^{kl} \right)(\vec{k}, \omega)$$



$$T(\vec{k}, \omega) = FT\left[T(\vec{x}, t) \right]$$

$$\hookrightarrow e_{\mu\nu} \propto T(\vec{k}, \omega) \Big|_{\vec{k} = \frac{\vec{x}}{r} \cdot \omega}$$