Gravitational waves

Gravitational waves have been another early prediction of GR. Even though it is a seamingly simple concept, there were doubts in the physics community about their existence. And it took until the 1950s that they have been established firmly (e.g. Feynman and the 'sticky beats').

Consider gravity in the weak field limit.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

In leader order, one has

$$R_{\mu\nu} = \frac{\partial}{\partial x^\nu} \Gamma^\lambda{}_{\lambda\mu} - \frac{\partial}{\partial x^\lambda} \Gamma^\lambda{}_{\mu\nu}$$

and the affine connection

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} \eta^{\lambda\rho} \left\{ \frac{\partial h_{\rho\nu}}{\partial x^\mu} + \frac{\partial h_{\rho\mu}}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x^\rho} \right]$$

In leading order, indices are raised/lowered with the Minkowski metric and one finds

$$R_{\mu\nu} = \frac{1}{2} \Big(\Box h_{\mu\nu} - \frac{\partial}{\partial x^\lambda \partial x^\mu} h^\lambda{}_\nu - \frac{\partial^2}{\partial x^\lambda \partial x^\nu} h^\lambda{}_\mu$$

$$+ \frac{\partial^2}{\partial x^\mu \partial x^\nu} h^\lambda{}_\lambda \Big)$$

And the Einstein equations

$$2 R_{\mu\nu} = -16\pi G \, S_{\mu\nu}$$

notice

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{\lambda}{}_{\lambda}$$

Furthermore, in leading order

$$\partial_\mu T^{\mu\nu} = 0 + O(h)$$

At this stage, we can use the gauge degree of freedom to simplify the equations of motion

$$g'_{\mu\nu} = \frac{dx^\alpha}{dx'^\mu}\frac{dx^\beta}{dx'^\nu} g_{\alpha\beta}$$

Since we are in the weak field limit, we can write

$$x'^\mu = x^\mu + \epsilon^\mu(x'^\mu)$$

where ϵ is considered O(h)

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \epsilon_\mu}{\partial x^\nu} - \frac{\partial \epsilon_\nu}{\partial x^\mu}$$

Also, in leading order, $T_{\mu\nu}$ is unchanged.

In the following, we will use the harmonic gauge, which implies

$$\Gamma^\lambda = g^{\mu\nu}\,\Gamma^\lambda{}_{\mu\nu} \overset{!}{=} 0$$

$$\hookrightarrow \frac{\partial}{\partial x^\mu} h^\mu{}_\nu = -\frac{1}{2}\frac{\partial}{\partial x^\nu} h^\mu{}_\mu$$

In this gauge, the Einstein equation reads

$$\Box h_{\mu\nu} = -16\pi G\, S_{\mu\nu}$$

This is exactly of the same form as the Maxwell equation in Lorentz-gauge.

$$h_{\mu\nu}(x,t) = 4G \int d^3x' \,\frac{S_{\mu\nu}(x', t-|x-x'|)}{|x-x'|}$$

At this point we would like to check explicitly that the harmonic gauge condition is fulfilled. And indeed it is, when energy momentum conservation is fulfilled:

$$\partial_\mu S^\mu{}_\nu = \frac{1}{2}\partial_\nu S^\mu{}_\mu$$

Notice also, that the solution is not unique. We can add any solution to the harmonic equation

$$\Box H_{\mu\nu} = 0$$

$\hookrightarrow$ $h_{\mu\nu} + H_{\mu\nu}$ is also a solution.

This change corresponds to a change of boundary conditions.

Plan waves:

We will see that the retarded solution will tend to plane waves away from the source. So let's discuss these first:

$$\text{Ansatz: } \quad h_{\mu\nu} = e_{\mu\nu} \exp\left(i k_\mu x^\mu\right) + \text{c.c.}$$

vacuum: $\Box h_{\mu\nu} = 0 \quad \rightarrow \quad k_\mu k^\mu = 0$

gauge: $\partial_\mu h^\mu{}_\nu = \frac{1}{2} \partial_\nu h^\mu{}_\mu \quad \rightarrow \quad k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\mu{}_\mu$

symmetry: $h_{\mu\nu} = h_{\nu\mu} \quad \rightarrow \quad e_{\mu\nu} = e_{\nu\mu}$

$e_{\mu\nu}$ is called the polarization tensor.

The metric has 10 degrees of freedom. 4 have been removed by the gauge fixing, which leaves 6.

However, after imposing the equation of motion, a residual gauge freedom shows up.

$$h_{\mu\nu} \to h_{\mu\nu} - \frac{\partial \epsilon_\mu}{\partial x^\nu} - \frac{\partial \epsilon_\nu}{\partial x^\mu}$$

We consider also plane waves for the gauge transformation:

$$\epsilon_\rho(x) = i\bar{\epsilon}_\mu \exp(ik_\mu x^\mu) + c.c.$$

This implies for the polarization tensor

$$e_{\mu\nu} \to e_{\mu\nu} + k_\mu \bar{\epsilon}_\nu + k_\nu \bar{\epsilon}_\mu$$

Notice that this residual gauge transformation never leaves the harmonic gauge (if we imply the EoM)

$$k_\mu (k^\mu \epsilon_\nu + k_\nu \epsilon^\mu) = \frac{1}{2} k_\nu (k^\rho \epsilon_\rho + k^\rho \epsilon_\rho)$$

$k^\rho k_\rho = 0$: $\quad k_\mu k_\nu \epsilon^\mu = k_\mu k_\nu \epsilon^\mu \quad \checkmark$

This gives another 4 gauge fixing conditions that can be used to remove 4 degrees of freedom and we are left with 2 dof.

For example, for a momentum along the z-direction:

$$k^\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix} \qquad (k_\mu k^\mu = 0)$$

the metric fluctuation can be brought to the form

$$h_{\mu\nu} \propto e^{ik_\mu x^\mu} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + c.c.$$

This fulfills the harmonic gauge constraints. The residual gauge freedom has been used to remove the trace and the temporal elements.

This is called the transverse traceless (TT) gauge.

$$h^\mu{}_\mu = 0 \; ; \quad h_{0\mu} = 0$$

h_{ij} transverse to k_i : $h_{ij} k_i = 0$

Energy-momentum of gravitational waves

In the Einstein equation there is a separation of matter in the energy-momentum tensor and gravity in the Ricci tensor.

$$G_{\mu\nu} = -8\pi G\, T_{\mu\nu}$$

While the notation of an energy momentum tensor of gravity in general does not exist, one can introduce this notion in the weak field limit:

$$G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + O(h^3) = -8\pi G\, T_{\mu\nu}$$

$$\hookrightarrow \quad G^{(1)}_{\mu\nu} = -8\pi G \left(T_{\mu\nu} + t_{\mu\nu} \right)$$

$$t_{\mu\nu} = \frac{1}{8\pi G} \left(G^{(2)}_{\mu\nu} + O(h^3) \right)$$

We want to interpret $G^{(2)}_{\mu\nu}$ as the energy-momentum tensor of the gravitational waves. Does this make sense?

Is this energy-momentum tensor conserved?

The Bianchi identity in the weak field limit implies

$$\partial_\nu \left(\eta^{\nu\lambda} \eta^{\mu\kappa} G_{\lambda\kappa}^{(1)} \right) = 0$$

This implies that

$$\partial_\nu \mathcal{T}^{\nu\mu} = 0$$

$$\mathcal{T}^{\mu\nu} = T^{\mu\nu} + t^{\mu\nu}$$

This is the energy-momentum tensor of matter+GWs.

This allows to construct conserved quantities like the energy-momentum vector:

$$P^\lambda = \int \mathcal{T}^{0\lambda} \, d^3x$$

which is related to the flux by

$$\frac{\partial}{\partial t} P^\lambda = \frac{\partial}{\partial t} \int_V \mathcal{T}^{0\lambda} \, d^3x = - \int_V \partial_i \mathcal{T}^{i\lambda} \, d^3x$$

$$= \int_{\delta V} T^{i\lambda} n_i \, dS$$ (energy-momentum flux)

Back to plane waves:

$$t^{\mu\nu} = \frac{1}{8\pi G} \left[R^{(2)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\lambda\rho} R_{\lambda\rho} \right]$$

This has to be evaluated now for our plane wave solution. When this is done there are terms that are proportional to

$$R^{(2)} \ni \exp\left(i \alpha k_\mu x^\mu\right) \qquad \alpha \in (0, \pm 2)$$

Any observable that is only sensitive to large volumes and large times

$$\Delta t, \Delta x \gg \frac{1}{k}$$

will be insensitive to these oscillating terms, and we will neglect them .

In the harmonic gauge, the remaining terms read

$$\langle R^{(1)}_{\mu\nu} \rangle = \frac{k_\mu k_\nu}{2} \left(e^{\lambda\rho *} e_{\lambda\rho} - \frac{1}{2} |e^\lambda{}_\lambda|^2 \right)$$

And in terms of helicities,
 it reads

$$\langle t_{\mu\nu} \rangle = \frac{k_\mu k_\nu}{16\pi G} \left(|h_\times|^2 + |h_+|^2 \right)$$