Some more comments on parallel transport

Notice due to parallel transport, one has to choose a path in order to compare vectors (tensors) at different points of the manifold.

In particular, if there is curvature, there is a path dependence and hence there is no generic way of comparing vectors at different points.

This has important implications for physics: there is no way to compare meaningfully velocities/spins of particles at different locations.

This means that interactions in almost all cases have to be local!

Seemingly non-local effects are 'mediated' by exchanging particles (e.g. photons)

-> causality structure

Bianchi identity

We have seen that $R_{\lambda\mu\nu\kappa}$

can be written as

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\mu \partial x^\kappa} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda \partial x^\kappa} \right]$$

$$+ g_{\rho\sigma} \Gamma^\rho_{\lambda\nu} \Gamma^\sigma_{\mu\kappa} - g_{\rho\sigma} \Gamma^\rho_{\lambda\kappa} \Gamma^\sigma_{\mu\nu}$$

So in the free falling system, we have

$$\left(g = \eta \big|_X , \quad \Gamma = 0 \big|_X \right)$$

$$\nabla_\eta R_{\lambda\mu\nu\kappa} \Big|_X = \frac{1}{2} \partial_\eta \left[\frac{\partial g_{\lambda\nu}}{\partial x^\mu \partial x^\kappa} + 3 \text{ perm} \right]$$

By permutation one then obtains:

$$\nabla_\eta R_{\lambda\mu\nu\kappa} + \nabla_\kappa R_{\lambda\mu\eta\nu} + \nabla_\nu R_{\lambda\mu\kappa\eta} \Big|_X = 0$$

Since this is a tensor relation,
this holds true in an arbitrary coordinate system everywhere.

$$\boxed{\nabla_\eta R_{\lambda\mu\nu\kappa} + \nabla_\kappa R_{\lambda\mu\eta\nu} + \nabla_\nu R_{\lambda\mu\kappa\eta} = 0}$$

Due to the fact that

$$(D_\mu D_\rho - D_\rho D_\mu) V^\lambda \equiv -R^\lambda{}_{\kappa\mu\rho} V^\kappa$$
$$\equiv [D_\mu, D_\rho] V^\lambda$$

$[A, B] = AB - BA$ commutator

The Bianchi identity can then be written:

$$[D_\alpha, [D_\beta, D_\gamma]] + [D_\gamma, [D_\alpha, D_\beta]]$$
$$+ [D_\beta, [D_\gamma, D_\alpha]] = 0$$

Contraction of the Bianchi identity

Contracting the Bianchi identity gives:

$$D_\eta R_{\mu\kappa} - D_\kappa R_{\mu\eta} + g^{\nu\lambda} D_\nu R_{\lambda\mu\kappa\eta} = 0$$

And once more:

$$D_\eta R - g^{\mu\kappa} D_\kappa R_{\mu\eta} - g^{\nu\lambda} D_\nu R_{\lambda\eta} = 0$$

$$g^{\kappa\nu} D_\kappa \left[R_{\nu\eta} - \frac{1}{2} g_{\nu\eta} R \right] = 0$$

This suggests the definition
(Einstein tensor, opposite trace tensor)

$$G_{\nu\eta} = R_{\nu\eta} - \frac{1}{2} g_{\nu\eta} R$$

and $g^{\kappa\nu} D_\kappa G_{\nu\eta} = 0$; $g^{\nu\eta} G_{\nu\eta} = -R$

Note: There is a sign difference
in the definition of the Riemann tensor
in Weinberg compared to Carroll

Weinberg: $X_{;\mu} = D_\mu X$

$$S^\lambda{}_{;\mu;\nu} - S^\lambda{}_{;\nu;\mu} \equiv R^\lambda{}_{\kappa\mu\nu}$$
$$= (D_\nu D_\mu - D_\mu D_\nu) S^\lambda$$
$$= -[D_\mu, D_\nu] S^\lambda$$

Hence for Weinberg, the curvature scalar
of a 2-sphere is negative!

$$ds^2 = L^2 (d\theta^2 + \sin\theta^2 d\varphi^2)$$
$$R = -\frac{2}{L^2}$$

And the relation with the Gauss curvature is

$$-\frac{R}{2} = \frac{\det M}{\det g} = K$$

Einstein equations

We have seen that the Riemann tensor contains second derivatives and probably is a good candidate for the kinetic term of the metric.

Hence a good guess for the action of general relativity is

$$\int d^4x \sqrt{g} \left\{ -\kappa R + \mathcal{L}_{matter} \right\}$$

$\mathcal{L}_{matter}$ contains the Lagrangian of the matter part, e.g.

$$\mathcal{L}_{matter} \ni -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad \eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$-\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi - V(\varphi)$$

And k is a constant with mass dimension 2

This action will lead to the equation of motion of the metric

$$\frac{\delta S}{\delta g^{\mu\nu}} - \frac{\partial}{\partial x^{\kappa}} \frac{\delta S}{\delta(\partial_\kappa g^{\mu\nu})} = 0$$

$$\hookrightarrow \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{1}{2\kappa} T_{\mu\nu}$$

with

$$T_{\mu\nu} = +2 \frac{1}{\sqrt{g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}$$

We have argued before that the Einstein tensor has to show up in this relation because the Bianchi identities imply

$$D_\mu G^{\mu\nu} = 0$$

while in general $\quad D_\mu R^{\mu\nu} \neq 0$

What remains to do is to fix the constant of proportionality κ.

We consider the non-relativistic limit with weak gravitational fields.

Remember the non-relativistic limit of the geodesic equation:

$$\frac{\partial^2 x^\mu}{(\partial\lambda)^2} + \Gamma^\mu_{\nu\kappa} \frac{\partial x^\nu}{\partial\lambda}\frac{\partial x^\kappa}{\partial\lambda} = 0$$

Non-relativistic motion means

$$\frac{\partial x^i}{\partial\lambda} \ll \frac{\partial x^0}{\partial\lambda} = \frac{\partial t}{\partial\lambda}$$

and weak fields

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{and} \quad h_{\mu\nu} \ll 1$$

$$\hookrightarrow \quad \frac{\partial^2 x^\mu}{(\partial\lambda)^2} + \Gamma^\mu_{00}\left(\frac{\partial t}{\partial\lambda}\right)^2 = 0$$

and also

$$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\nu}\left[\frac{\partial h_{0\nu}}{\partial t} + \frac{\partial h_{\nu 0}}{\partial t} - \frac{\partial h_{00}}{\partial x^\nu}\right]$$

If the particles are slow, then also the generated gravitational fields will change slowly:

$$\frac{\partial h_{\mu\nu}}{\partial t} \ll \frac{\partial h_{\mu\nu}}{\partial x^i}$$

$$\hookrightarrow \quad \Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu i}\frac{\partial h_{00}}{\partial x^i}$$

$$\hookrightarrow \quad \frac{\partial^2 t}{(\partial\lambda)^2} = 0 \quad \Rightarrow \quad t \propto \lambda$$

$$\frac{\partial^2 \vec{x}}{(\partial\lambda)^2} = \left(\frac{\partial t}{\partial\lambda}\right)^2 \frac{1}{2} \vec{\nabla} h_{00}$$

$$\hookrightarrow \quad \frac{\partial^2 \vec{x}}{(\partial t)^2} = \frac{1}{2} \vec{\nabla} h_{00}$$

Comparing to Newtons law, we find that

$$\frac{\partial^2 \vec{x}}{(\partial t)^2} = -\nabla \Phi$$

$$\hookrightarrow \quad h_{00} = -2\Phi \quad , \quad g_{00} = -(1+2\Phi)$$

In order to find the equivalent of the Poisson equation:

$$\Delta \Phi = 4\pi G \rho$$

where ρ is the energy density and G is the Newton constant we have to consider the Einstein equation.

Note: The scalar from the Einstein tensor is

$$g^{\mu\nu} G_{\mu\nu} = -R$$

and has nothing to do with G.

For non-relativistic matter, one has

$$T^{00} \simeq \rho$$

and hence the Einstein equation should read

$$\Delta h_{00} = -8\pi G \, T_{00}$$

as compared to

$$G_{00} = -\frac{1}{2k} T_{00}$$

So we have to calculate G_{00} in the non-realtivistic limit.

In the non-relativistic limit, one has

$$T_{i0}, T_{ij} \ll T_{00}$$

Due to the Einstein equation this directly implies

$$G_{i0}, G_{ij} \ll G_{00}$$

and

$$G_{ij} = R_{ij} - \frac{1}{2}\eta_{ij} R \simeq 0$$

$$\rightarrow \quad R_{ij} = \frac{1}{2}\eta_{ij} R$$

$$R = \eta^{\mu\nu} R_{\mu\nu} = \left(-R_{00} + \frac{3}{2} R\right) \Rightarrow R = 2 R_{00}$$

Finally:

$$G_{00} = R_{00} - \frac{1}{2}\eta_{00} R = 2 R_{00}$$

By definition:

$$\Gamma = O(h)$$

$$R_{\mu\nu\lambda\kappa} = \frac{1}{2}\left[\frac{\partial h_{\mu\lambda}}{\partial x^{\nu}\partial x^{\kappa}} + 3\,\text{perm} \right] + O(\Gamma^2)$$

For static fields:

$$R_{\mu 0 \lambda 0} = \frac{1}{2}\frac{\partial h_{00}}{\partial x^{\mu}\partial x^{\lambda}}$$

$$R_{00} = \eta^{\mu\lambda} R_{\mu 0 \lambda 0} = \frac{1}{2}\Delta h_{00}$$

$\hookrightarrow$ $G_{00} = \Delta h_{00}$

$\hookrightarrow$ $k = \frac{1}{16\pi G} = \frac{M_{Pe}^2}{2}$ reduced Planck mass

$\hookrightarrow$ $$\boxed{G_{\mu\nu} = -8\pi G\, T_{\mu\nu}}$$

$$M_{Pe} \approx 10^{18}\ \text{GeV}$$