Minkowski space has $R^{\lambda}{}_{\mu\nu\kappa} = 0$

so it will be true in any representation of flat space.

For example in spherical coordinates:

$$ds^2 = dr^2 + r^2 \left(d\theta^2 + \cos^2\theta \, d\varphi^2\right) - dt^2$$

$$ds^2 = dx^\mu dx^\nu g_{\mu\nu} \qquad (\mu = t, r, \theta, \varphi)$$

$\Gamma^{\lambda}{}_{\mu\nu} \neq 0$ because of the pseudo forces

but $R^{\lambda}{}_{\mu\nu\kappa} = 0$ because it is a tensor

Is the opposite also true? If $R^{\lambda}{}_{\mu\nu\kappa} = 0$

does this mean that the space is (locally) equivalent to Minkowski space?

Actually, if $R^{\lambda}{}_{\mu\nu\kappa}=0$ and the signature of the metric is the one of Minkowski, this is in fact true.

Let's start in an arbitrary point X.

$$\exists\, d^{\mu}{}_{\alpha} \qquad g_{\mu\nu}(X)\, d^{\mu}{}_{\alpha} d^{\nu}{}_{\beta} = \eta_{\alpha\beta}$$

These matrices can be enhanced to fields via parallel transport (remember $R^{\lambda}{}_{\mu\nu\kappa} = 0$)

$$\nabla_k d^{\mu}{}_{\alpha}(x) = \partial_k d^{\mu}{}_{\alpha} + \Gamma^{\mu}{}_{k\nu} d^{\nu}{}_{\alpha} = 0$$

(notice that alpha is not a GR index, it just enumerates the different vectors that achieve the Minkowski metric)

Then one has:

$$\partial_k \big(\underbrace{g_{\mu\nu}\, d^{\mu}{}_{\alpha} d^{\nu}{}_{\beta}}_{\text{16 scalars in GR}} \big) = 0$$

because $\nabla d = 0$
$\nabla g = 0$

$$\Rightarrow \quad g_{\mu\nu}\, d^{\mu}{}_{\alpha} d^{\nu}{}_{\beta} = \eta_{\alpha\beta} \quad \text{everywhere in the neighborhood}$$

Curvature:

In order to describe the dynamics of the metric, we need a 'kinetic term' for the metric.

Usually, kinetic terms contain second derivatives or two simple derivatives, e.g.

$$\mathcal{L} \ni \underbrace{\tfrac{1}{2} m \dot{x}^2}_{\text{kin.}} - V(x)$$

$$\hookrightarrow \quad \ddot{x} \propto -\frac{\partial V}{\partial x}$$

$$\mathcal{L} \ni \underbrace{-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}}_{\text{kin}} + A_\mu J^\mu$$

$$\hookrightarrow \quad \partial_\mu F^{\mu\nu} = J^\mu$$

$$\mathcal{L} \ni \tfrac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi + \tfrac{1}{2} m^2 \varphi^2 - V(\varphi)$$

$$\hookrightarrow \quad \Box \varphi + m^2 \varphi = \frac{\partial V}{\partial \varphi}$$

Hence we would like to construct a term involving second derivatives of the metric in a covariant way:

$$\nabla \ni \partial \cdot + \Gamma \qquad \Gamma = g \frac{\partial g}{\partial x} \cdots$$

$$O(\Gamma^2), \quad O(\partial \Gamma) \propto \frac{\partial^2 g}{\partial x^2}, \left(\frac{\partial g}{\partial x}\right)^2$$

Therefore we consider:

$$\nabla_\mu \nabla_\nu S^\kappa = \nabla_\mu \left(\partial_\nu S^\kappa + \Gamma^\kappa_{\ \nu\rho} S^\rho \right)$$

$$= \partial_\mu \partial_\nu S^\kappa + (\partial_\mu \Gamma^\kappa_{\ \nu\rho}) S^\rho + \Gamma^\kappa_{\ \nu\rho} \partial_\mu S^\rho$$

$$- \Gamma^\alpha_{\ \nu\mu} \partial_\alpha S^\kappa + \Gamma^\kappa_{\ \mu\alpha} \partial_\nu S^\alpha$$

$$- \Gamma^\alpha_{\ \mu\nu} \Gamma^\kappa_{\ \alpha\rho} S^\rho + \Gamma^\kappa_{\ \mu\alpha} \Gamma^\alpha_{\ \nu\rho} S^\rho$$

Notice that the zeroth and first order terms are symmetric in μ and ν.

In order to eliminate them, we can form the antisymmetric combination again:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S^\kappa$$

$$= (\partial_\mu \Gamma^\kappa_{\ \nu\rho} - \partial_\nu \Gamma^\kappa_{\ \mu\rho}) S^\rho$$

$$+ (\Gamma^\kappa_{\ \mu\lambda} \Gamma^\lambda_{\ \nu\rho} - \Gamma^\kappa_{\ \nu\lambda} \Gamma^\lambda_{\ \mu\rho}) S^\rho$$

$$\equiv - R^\kappa_{\ \rho\mu\nu} S^\rho = R^\kappa_{\ \rho\nu\mu} S^\rho$$

Where $R^\kappa_{\ \rho\mu\nu}$ is again the Riemann-Christoffel tensor.

Comments:

Notice that since $R^{\mu}{}_{\nu\lambda\kappa}$ is a tensor, it cannot be made to vanish by a coordinate transformation.

$$g_{\mu\nu}(y) = g_{0\,\mu\nu}(x) + (y^{\lambda} - x^{\lambda})\,\partial_{\lambda} g_{0\,\mu\nu}(x) + \frac{1}{2}(y^{\kappa} - x^{\kappa})(y^{\lambda} - x^{\lambda})\,\partial_{\kappa}\partial_{\lambda} g_{0\,\mu\nu}(x) + \dots$$

In every point, g0 can be made the Minkowski metric and ∂_{λ} g0 can be transformed away, but the second derivatives can only put partially in a more convenient form, because these are the degrees of freedom that enter the Riemann tensor, and it cannot be be removed fully (if the space is non-flat).

Notice that the construction of R via the commutator of the covariant derivative does not resort to parallel transport, S^{λ} is just any vector/field.

Generalizations:

So we have seen that

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S^\lambda = -R^\lambda{}_{\rho\mu\nu} S^\rho$$

This can be generalized for the commutator acting on different tensor structures:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \varphi = 0$$

For a covariant vector one finds:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) S_\lambda = +R^\rho{}_{\lambda\mu\nu} S_\rho$$

A similar formula holds for any tensor, e.g.

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T^\lambda{}_\kappa = -R^\lambda{}_{\rho\mu\nu} T^\rho{}_\kappa + R^\rho{}_{\kappa\mu\nu} T^\lambda{}_\rho$$

Symmetries:

In order to study the symmetries under exchange of indices one has to construct the fully covariant form:

$$R_{\lambda\kappa\mu\nu} = g_{\lambda\rho} R^\rho{}_{\kappa\mu\nu}$$

The fully covariant form can be evaluated to be:

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\mu \partial x^\kappa} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right]$$

$$+ g_{\eta\sigma}\left[\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu} \right]$$

The following properties are then more evident:

A) symmetry: $\quad R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu}$

$(12 \leftrightarrow 34)$

B) antisymmetry: $\quad R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu}$

$(3) \leftrightarrow (4)$; $(1) \leftrightarrow (2)$

$$= + R_{\mu\lambda\kappa\nu}$$

C) cyclicity:

(234)

$$R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0$$

In general this tensor would have $D^4 = 256$ arbitrary entries.

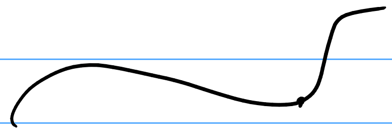
If all the symmetries are imposed, one ends up with

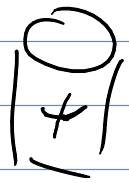
$\frac{1}{12} D^2 (D^2 - 1)$ (see e.g. Weinberg)

So the number of independent degrees of freedom is

$$\begin{aligned} D=4 &: \quad 20 \\ D=3 &: \quad 6 \\ D=2 &: \quad 1 \\ D=1 &: \quad 0 \end{aligned}$$

In particular, in 1D there is no 'curvature'. A line is always flat and curvature requires 2 dimensions





Contractions:

Due to the symmetries, $R^{\lambda}{}_{\kappa\mu\nu}$ only allows one non-trivial contraction, the Ricci tensor:

$$R_{\mu\kappa} = R^{\lambda}{}_{\mu\lambda\kappa} \qquad \left(D(D+1)/2 = 10 \text{ dof} \right)$$

The final contraction gives the curvature (Ricci) scalar

$$R = g^{\mu\nu} R_{\mu\nu}$$

Counting the degrees of freedom suggests that in 2D, the Riemann tensor might be expressed in terms of the Ricci scalar! (both have 1 d.o.f.)

$$R_{\lambda\kappa\mu\nu} = \frac{1}{2}\left(g_{\lambda\mu}g_{\kappa\nu} - g_{\lambda\nu}g_{\kappa\mu}\right)R$$