General theory of relativity

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outline:
- special relativity
- Lagrangian formalism
- manifolds and tensors
- physics in curved spacetime
- curvature

applications:
- black holes
- gravitational waves
- cosmology
- Penrose diagrams

special relativity

Before the advent of SR the laws of physics were believed to be invariant under Galilean transformantions:

$$\vec{x}' = R\vec{x} + \vec{v}t + \vec{a} \qquad R \in SO(3)$$
$$t' = t + \tau$$

For example

$$\vec{F} = m \cdot \vec{a} = m \frac{d^2\vec{x}}{dt^2} \to \vec{F}' = R\vec{F}$$

$$\vec{F} = G \frac{mM \, (\vec{x} - \vec{x}_M)}{|\vec{x} - \vec{x}_M|^3} \to \vec{F}' = R\vec{F}$$

mass M at $\vec{x}_M$.

On the other hand, there are many coordinate transformations that don't leave the laws unchanged, e.g. acceleration or rotation.

So there seem to be some class of 'special frames' without pseudo forces.
(see Mach's principle)

However, there are two facts
challenging this picture:

- Maxwell's equations are not invariant under Galilean transformations
- Michelson-Morley

Lorentz transformations

We start from the observation that the speed of light is the same in different frames, even if they are moving with a relative velocity.

path of light:

$$t(\tau), \vec{x}(\tau) \Rightarrow x^\mu(\tau)$$

$$\mu = 0, 1, 2, 3$$

$$\uparrow$$

$$t$$

$$-c^2 dt^2 + d\vec{x}^2 = ds^2 \overset{!}{=} 0$$

[natural units are c = k = ħ = 1]

[notice the minus sign, in GR the convention is to put the minus sign in front of the dt term]

We look for transformations that leave ds^2 invariant and are linear

$$dx'^{\mu} = \sum_{\nu} \Lambda^{\mu}{}_{\nu}\, dx^{\nu}$$
$$\equiv \Lambda^{\mu}{}_{\nu}\, dx^{\nu}$$

[Einstein's sum convention: indices that appear twice are summed over and there must be pairs of super- and sub-scripts]

One way of writing the line element using vectors is to introduce the metric

$$ds^2 = \eta_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$$

with $\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

This is the Minkowski metric.

Lorentz transformations should leave the line element unchanged which leads to

$$\Lambda^{\alpha}{}_{\gamma}\, \Lambda^{\beta}{}_{\delta}\, \eta_{\alpha\beta} = \eta_{\gamma\delta}$$

proper Lorentz transformations:

$\Lambda^0{}_0 \geq 1$ and $\det \Lambda = +1$

The Lorentz transformations contain rotations:

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} 1 & \\ & R_{ij} \end{pmatrix} \qquad R_{ij} \in SO(3)$$

$$\rightarrow \quad R^T \mathbb{1}_{3\times 3} \cdot R = \mathbb{1}_{3\times 3}$$

On the other hand

$$\vec{x} \rightarrow \vec{x}' = \vec{x} + \vec{v}t$$

is not part of the Lorentz transformations.
Instead we have boosts, e.g.

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \cosh\varphi & \sinh\varphi & & \\ \sinh\varphi & \cosh\varphi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In particular, velocities are not additive but

$$\Lambda(\varphi_1) \cdot \Lambda(\varphi_2) = \Lambda(\varphi_1 + \varphi_2)$$

φ is the boost parameter.

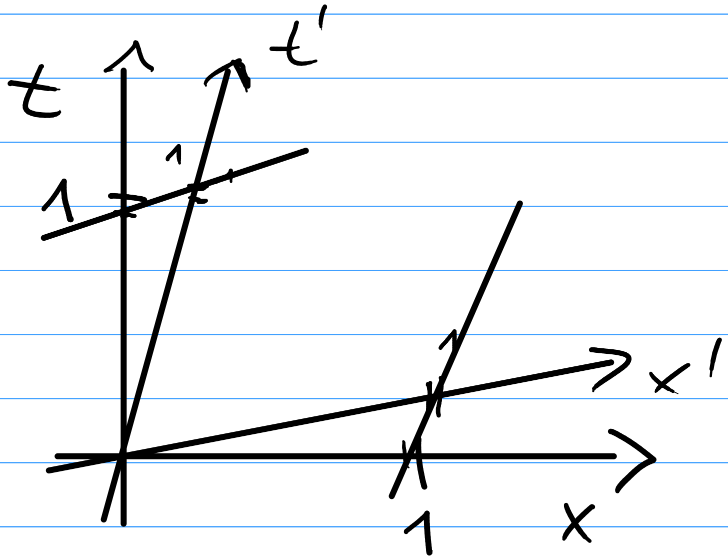
[check using

$$\cosh a \cosh b + \sinh a \sinh b = \cosh(a+b)$$
$$\cosh a \sinh b + \sinh a \cosh b = \sinh(a+b$$

A more familiar way to write this is using the parameters

$$\sinh\varphi = v\gamma \quad ; \quad \cosh\varphi = \gamma$$

$$\cosh^2 - \sinh^2 = 1 \Rightarrow \gamma^2 = \frac{1}{1-v^2}$$



time dilatations

$\sinh -\varphi = -\sinh\varphi$
$\cosh(-\varphi) = +\cosh\varphi$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh & -\sinh \\ -\sinh & \cosh \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

An object that is at rest in the primed frame

$$x' = 0 \Leftrightarrow x = \tanh\varphi \cdot t$$
$$= v.t$$

So the path of the object is

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ v \end{pmatrix}\tau \qquad \begin{pmatrix} t' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} 1 \\ v \end{pmatrix}\tau$$

$$= \begin{pmatrix} \tau/\gamma \\ 0 \end{pmatrix}$$