# QCD and Monte Carlo simulation

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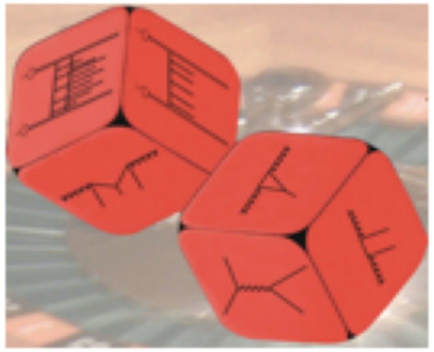
**http://www.desy.de/~jung/qcd_and_mc_2016**



# QCD AND MONTE CARLOS (2016)

- Lecture notes
- Lectures
  18. Oct 2016 Lecture 1
  20. Oct 2016 Lecture 2

- Exercises
  - Root in 5 seconds

- Exercise 1 text sheet
  - Exercise 1 template

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**QCD and Monte Carlos**
**Lecture Course**
**Winter 2016**
**Hannes Jung (DESY)**
**IFJ (Polish Academy of Science)**
**Cracow**

The lecture course is intended for master students, PhD students and postdocs. It covers a basic introduction to the QCD evolution equations. A basic introduction to Monte Carlo methods will be given, and these methods will be applied to calculate the evolution of parton densities.

During the course we will learn, how to write a small program to integrate cross sections. We will apply Monte Carlo methods to solving the DGLAP evolution equation and to determine the transverse momentum distribution of partons inside the proton (TMD).

Lectures will be on
Monday 18. Oct 16:00 - 17:00
Tuesday 20. Oct 10:00 - 11:00

# Outline of the lectures

- 18. Oct   Intro to Monte Carlo techniques and evolution equation
- 20. Oct   Solution of evolution equation with MC technique
- 25. Oct   Soft gluon effects and $p_T$ distribution
- 27. Oct   Color, coherence and parton shower

- Exercises

# recap: Randomness tests

- Congruential generator
- RANLUX

M. Lüscher, A portable high-quality random number generator for lattice field theory simulations, Computer Physics Communications 79 (1994) 100
http://luscher.web.cern.ch/luscher/ranlux/index.html

➔ RANLUX much more sophisticated. Developed and used for QCD lattice calcs

**Exercise**

# recap: Period of random number generator

- RANLUX: period ~ $10^{171}$
- LHC events:
  - $\sigma_{tot} = 100\, mb$
  - $\int L\, dt = 100\ fb^{-1}$
  - Nr of events: $\sim 10^{14}$
  - random numbers per event:
    - $\phi, \eta, p, m$ per particle
    - $\sim 2000$ particles/event

    $\rightarrow\ \sim 8\, 10^3 \sim 10^4$ random numbers/event

    $\rightarrow\ \sim 10^{14}\, 10^4 = 10^{18}$ random number for LHC simulation needed
- RANLUX is still good enough for this !

# recap: but more important are randomness tests

- statistical test (uniformity)
- serial tests
- sequence up – sequence down test
- gap – test
- random walk test
- and ...

# recap: Importance Sampling

- MC calculations most efficient for small weight fluctuations:

  $f(x)dx \to f(x)\, dG(x)/g(x)$

- chose point according to g(x) instead of uniformly
- f is divided by $g(x) = dG(x)/dx$
- **generate *x* according to:**

$$R \int_a^b g(x')dx' = \int_a^x g(x')dx'$$

- relevant variance is now V(f/g):

  small if g(x) ~ f(x)

- how-to get g(x)
  (1) g(x) is probability: g(x) > 0 and ∫dG(x) = 1
  (2) integral ∫dG(x) is known analytically
  (3) *G(x)* can be inverted (solved for *x*)
  (4) f(x)/g(x) is nearly constant, so that *V(f/g)* is small compared to V(f)

# recap: Monte Carlo integration versus others

## One dimensional quadrature

$$I = \int f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

- **Monte Carlo: Hit & Miss**

  $w$ =1 and $x_i$ chosen randomly

- **Trapezoidal Rule:**

  approximate integral in sub-interval by area of trapezoid below (above) curve

- **Simpson quadrature**

  approximate by parabola

- **Gauss quadrature**

  approximate by higher order polynomial

| method | err (1d) | error |
|---|---|---|
| MC | $n^{-1/2}$ | $n^{-1/2}$ |
| Trapez | $n^{-2}$ | $n^{-2/d}$ |
| Simpson | $n^{-4}$ | $n^{-4/d}$ |
| Gauss | $n^{-2m+1}$ | $n^{-(2m-1)/d}$ |

# recap: QCD and Monte Carlo simulation

## Part 2: Evolution equations

# recap: Probing the structure of matter

- virtual photons (in analogy to optics)
- BUT also:
  - ➔ Z/W exchange
  - ➔ Jet production
  - ➔ Heavy quarks
- Example Deep Inelastic Scattering:
  - Kinematics:

$$s = (e+p)^2$$
$$q = e - e'$$
$$Q^2 = -q^2$$
$$y = \frac{q.p}{e.p}$$
$$W^2 = (q+p)^2$$
$$x = \frac{Q^2}{2p.q}$$

- in p-rest frame:

$$p = (M, 0)$$
$$\nu = \frac{p.q}{M} = \frac{M q_0}{M} = E - E'$$

- using

$$W^2 = (q+p)^2 = M^2 + 2q.p - Q^2$$

$$\nu = \frac{Q^2 + W^2 - M^2}{2M}$$

# recap: Inelastic Scattering: QPM

Ellis,Webber, Stirling, p 90 ff

- Infinite momentum frame: $p^\mu = (P, 0, 0, P)$ with $P \gg M$
- Virtual photon scatters off point-like quark which moves parallel (collinear) to proton, with momentum fraction $p_q^\mu = \xi p^\mu$
- Using DIS variables gives for $eq \to eq$
- giving

$$|M|^2 = \frac{2e_q^2(4\pi\alpha)^2\hat{s}^2}{Q^4}\left(1+(1-y)^2\right)$$

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left(1+(1-y)^2\right)$$

- Using mass shell condition for outgoing quark gives (with $\int_0^1 dx\delta(x-\xi) = 1$ )

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left(1+(1-y)^2\right)\frac{1}{2}e_q^2\delta(x-\xi)$$

- compare this with formula for DIS

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1+(1-y)^2\right)F_1 + \frac{1-y}{x}(F_2 - 2xF_1)\right]$$

# recap: Inelastic Scattering QPM

- Simple model with

$$\tilde{F}_2 = x e_q^2 \delta(x-\xi) = 2x\tilde{F}_1$$

- BUT structure function is a distribution.
  $F_2$ is a function of *x:* **scaling, no $Q^2$ dependence**

- $q(\xi)d\xi$ is probability to find q with momentum fraction $\xi....\xi + d\xi$

$$F_2(x) = 2xF_1(x) = \sum_{q,\bar{q}} \int_0^1 d\xi q(\xi) x e_q^2 \delta(x-\xi) = \sum_{q,\bar{q}} e_q^2 x q(x)$$

- Proton structure function is:

$$F_2^{em} = x\left[\frac{4}{9}\left(u(x)+\bar{u}(x)+c(x)+\bar{c}(x)\right)+\frac{1}{9}\left(d(x)+\bar{d}(x)+s(x)+\bar{s}(x)\right)\right]$$

# Structure functions from HERA

- Proton structure function does not depend on $Q^2$ for large $x$
- $F_2$ scales ...
- Quarks are pointlike constituents of proton
- BUT things change at smaller $x$..... and smaller $Q^2$

# Deeper look to x-section:
# separate leptonic from hadronic part

# Higher order corrections to DIS

- lowest order: $e + q \to e' + q' \quad \mathcal{O}(\alpha_s^0)$

$$\frac{d\sigma}{dy dQ^2} = \frac{4\pi\alpha^2}{Q^4}\frac{x}{y}\left(1 + (1-y)^2\right)\frac{1}{2}e_q^2\delta(x-\xi)$$

- higher order: $e + q \to e' + q' + g, \quad e + g \to e' + q + \bar{q} \quad \mathcal{O}(\alpha_s^1)$
- factorize electromagnetic vertex or calculate full $2 \to 3$ process
- use Weizsäcker(Z. Phys 88, 612 (1934)) -Williams (Phys Rev 45, 729 (1934)) (or Equivalent Photon(Budnev Phys Rep C15 , 181 (1974))) Approximation:

**from:**

$$\frac{d\sigma}{dy dQ^2} = \frac{\alpha}{2\pi}\frac{1}{yQ^2}\left(1 + (1-y)^2\right)\frac{4\pi^2\alpha}{Q^2}e_q^2 x\delta(x-\xi)$$

**obtain:**

$$\frac{d\sigma}{dy dQ^2} = F_{\gamma/e}(y, Q^2)\,\sigma(\gamma^* q \to q')$$

Isolate dominant parts

in the matrix elements:

region of small $k_t$ !!!

# Higher order corrections to DIS

- lowest order: $e + q \to e' + q' \quad \mathcal{O}(\alpha_s^0)$
- higher order: $e + q \to e' + q' + g, \quad e + g \to e' + q + \bar{q} \quad \mathcal{O}(\alpha_s^1)$
  - What is the dominant part of the x-section ?
    - Investigate full x-section of QCDC and BGF
    - dominant part comes from small transverse momenta ...
    - rewrite x-section in terms of $k_\perp$
    - use small $t$ limit:

$$\frac{d\sigma}{dk_\perp^2} = \frac{d\sigma}{dt}\frac{1}{(1-z)} = \frac{1}{(1-z)}\frac{1}{F}dLips|ME|^2$$

$$= \frac{1}{(1-z)}\frac{1}{16\pi}\frac{1}{\hat{s}+Q^2}\frac{1}{\hat{s}}|ME|^2$$

# Kinematics

$$\hat{s} = (p_1 + p_2)^2 = Q^2 \frac{1-z}{z}$$

$$\hat{t} = k^2 = (p_1 - p_3)^2$$

$$\hat{u} = (p_2 - p_3)^2$$

**Define:**

$$\xi = \frac{p_2 k}{p_1 p_2} = 1 - \frac{p_2 p_3}{p_1 p_2}$$

$$z = \frac{Q^2}{2 p_1 p_2}$$

$$x_{bj} = z\xi$$

Using $s+t+u = -Q^2$ gives:

$$k_\perp^2 = \frac{\hat{t}\hat{u}\hat{s}}{(\hat{s}+Q^2)^2}$$

- and for

$$\hat{t} \ll \hat{s}$$

$$k_\perp^2 = \frac{-\hat{t}\hat{s}}{\hat{s}+Q^2} = -t\,(1-z)$$

# QCDC - contribution

- use matrix element and perform small $t$ approximation (where the ME is largest !):

$$
\begin{aligned}
\hat{s} &= (p_1 + p_2)^2 \\
&= Q^2 \frac{1-z}{z} \\
u &= -Q^2 - s - t \\
&\rightsquigarrow -Q^2 - s \\
z &= \frac{Q^2}{2p_1 p_2}
\end{aligned}
$$

# QCDC - contribution

$$|M|^2 = 32\pi^2 \left(e_q^2 \alpha \alpha_s\right) \frac{4}{3} \left[ \frac{-\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}} \right]$$

$$= 32\pi^2 \left(e_q^2 \alpha \alpha_s\right) \frac{4}{3} \frac{-1}{t} \left[ \frac{Q^2(1+z^2)}{z(1-z)} + \cdots \right]$$

integration over $k_\perp$ generates $log$ BUT what is the lower and upper limit ?

$$k_{Tmax}^2 = \frac{\hat{s}}{4} = \frac{Q^2(1-z)}{4z}$$

$$\frac{d\sigma}{dk_\perp^2} = \hat{\sigma}_0 e_q^2 \frac{\alpha_s}{2\pi} \frac{1}{k_\perp^2} \left[ P_{qq}(z) + \cdots \right]$$

$$P_{qq}(z) = \frac{4}{3} \frac{1+z^2}{1-z} \qquad \hat{\sigma}_0 = \frac{4\pi^2 \alpha}{\hat{s}}$$

$$\sigma^{QCDC} = \hat{\sigma}_0 e_q^2 \frac{\alpha_s}{2\pi} \left[ P_{qq}(z) \log\left( \frac{Q^2(1-z)}{\chi^2 z} \right) + \cdots \right]$$

# Correction to cross section

- Connect with $F_2$:

$$\sigma^{\gamma * p} = \frac{4\pi^2\alpha}{Q^2}\left(F_2(x,Q^2) - F_L(x,Q^2)\right) \sim \frac{4\pi^2\alpha}{Q^2} F_2(x,Q^2) = \frac{4\pi^2\alpha}{2qP}\frac{F_2(x,Q^2)}{x}$$

$$\sigma^{\gamma * p} = \sigma_0 \frac{F_2(x,Q^2)}{x}$$

$$\sigma^{QPM} = \sigma_0 e_q^2 \delta\left(1 - \frac{x}{\xi}\right)$$

$$\sigma^{QCDC} = \hat{\sigma}_0 e_q^2 \otimes P_q(z) \otimes \log \ldots$$

$\xi$ is parton momentum fraction

$$\sigma_0 = \frac{4\pi^2\alpha}{2qP}$$

# QCDC contribution to $F_2$

$$\frac{F_2}{x} =$$

again divergency for $k_\perp \to 0$ **or** $\chi \to 0$

- 

$$\begin{aligned}\frac{F_2}{x} &= \sum e_q^2 \int \frac{d\xi}{\xi} f_q(\xi) \left[ \delta\left(1 - \frac{x}{\xi}\right) + \right. \\ & \left. \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \left[ \log\left(\frac{Q^2}{\chi^2}\right) + \log\left(\frac{1-z}{z}\right) + \ldots \right] + C_q(z, ..) \right]\end{aligned}$$

# Boson gluon fusion

- use BGF matrix element and perform small $t$ approximation (where the ME is largest !):

$$\begin{aligned}
\hat{s} &= (p_1 + p_2)^2 \\
&= Q^2 \frac{1-z}{z} \\
u &= -Q^2 - s - t \\
&\rightsquigarrow -Q^2 - s \\
z &= \frac{Q^2}{2 p_1 p_2}
\end{aligned}$$

# Boson gluon fusion

$$|M|^2 = 32\pi^2 \left(e_q^2 \alpha \alpha_s\right) \frac{1}{2} \left[\frac{\hat{u}}{\hat{t}} + \frac{\hat{t}}{\hat{u}} - \frac{2\hat{s}Q^2}{\hat{t}\hat{u}}\right]$$

$$\frac{d\sigma}{dk_\perp^2} = \hat{\sigma}_0 e_q^2 \frac{\alpha_s}{2\pi} \frac{1}{k_\perp^2} \left[P_{qg}(z) + \cdots\right]$$

$$P_{qg}(z) = \frac{1}{2}\left(z^2 + (1-z)^2\right)$$

- integration over $k_t$ generates *log*, BUT what is the lower limit

$$\sigma^{BGF} = \hat{\sigma}_0 e_q^2 \frac{\alpha_s}{2\pi} \left[P_{qg}(z) \log\left(\frac{Q^2(1-z)}{\chi^2 z}\right) + \cdots\right]$$

# BGF contribution to $F_2$

$$\frac{F_2}{x} =$$

again divergency for $k_\perp \to 0$ or $\chi \to 0$

$$\begin{aligned} \frac{F_2}{x} &= \sum e_q^2 \int \frac{dx_2}{x_2} g(x_2) \\ & \frac{\alpha_s}{2\pi} \left( P_{qg}\left(\frac{x}{x_2}\right) \left[ \log\left(\frac{Q^2}{\chi^2}\right) + \log\left(\frac{1-z}{z}\right) + \ldots \right] + C_g(z,\ldots) \right) \end{aligned}$$

# Adding up everything

$$\sigma^{\gamma * p} \sim \frac{4\pi^2\alpha}{Q^2} F_2(x,Q^2) = \frac{4\pi^2\alpha}{ys}\frac{F_2(x,Q^2)}{x}$$

$$\sigma^{\gamma * p} = \sigma_0 \frac{F_2(x,Q^2)}{x}$$

$\xi$ is parton momentum fraction

$$\sigma_0 = \frac{4\pi^2\alpha}{2qP}$$

- Connect with $F_2$:

$$\sigma^{QPM} = \sigma_0 e_q^2 \delta\left(1 - \frac{x}{\xi}\right)$$

$$\sigma^{QCDC} = \hat{\sigma}_0 e_q^2 \otimes P_q(z) \otimes \log \ldots$$

$$\sigma^{BGF} = \hat{\sigma}_0 e_q^2 \otimes P_g(z) \otimes \log \ldots$$

# Collinear factorization (part 1)

- bare distributions $q_0(x)$ are not measurable (like the bare charges .... )

$$F_2 = x \sum e_q^2 \left[ q_0(x) + \int \frac{d\xi}{\xi} q_0(x) \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \log\left(\frac{Q^2}{\chi^2}\right) + C_q(z,..) \right]$$

- collinear singularities are absorbed into this bare distributions at a factorization scale $\mu^2 \gg \chi^2$ , defining renormalized distributions

$$q_i(x,\mu^2) = q_i^0(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} \left[ q_i^0(\xi) P_{qq}\left(\frac{x}{\xi}\right) \log\left(\frac{\mu^2}{\chi^2}\right) + C_q\left(\frac{x}{\xi}\right) \right] + \ldots$$

- now $F_2$ becomes:

$$F_2 = x \sum e_q^2 \int \frac{d\xi}{\xi} q(\xi,\mu^2) \left[ \delta\left(1 - \frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \log\left(\frac{Q^2}{\mu^2}\right) + C \right]$$

- separating or factorizing the long distance contributions to structure functions is a fundamental property of the theory
- factorization provides a description for dealing with the logarithmic singularities, there is arbitrariness in how the finite (non-logarithmic) parts are treated.

# Collinear factorization: DGLAP

- start from:

$$F_2 = x \sum e_q^2 \int \frac{d\xi}{\xi} q(\xi, \mu^2) \left[ \delta\left(1 - \frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \log\left(\frac{Q^2}{\mu^2}\right) + C \right]$$

- and require that $\frac{\partial F_2}{\partial \mu^2} = 0$

$$\begin{aligned}
\frac{\delta F_2}{\delta \mu^2} &= \int \frac{d\xi}{\xi} \left( \frac{\partial q(\xi,\mu^2)}{\partial \mu^2} \left[ \delta\left(1 - \frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \log\left(\frac{Q^2}{\mu^2}\right) \right] \right. \\
&\quad \left. + q(\xi,\mu^2) \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \frac{\partial}{\partial \mu^2} \left[ \log Q^2 - \log \mu^2 \right] \right) \\
&= \frac{\partial q(x,\mu^2)}{\partial \mu^2} + \int \frac{d\xi}{\xi} \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \log \frac{Q^2}{\mu^2} \frac{\partial q(\xi,\mu^2)}{\partial \mu^2} \\
&\quad + \int \frac{d\xi}{\xi} q(\xi,\mu^2) \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \left( -\frac{1}{\mu^2} \right)
\end{aligned}$$

Neglect 2nd term, because is of order ${\alpha_s}^2$

leading to:

$$\frac{dq_i(x,\mu^2)}{d \log \mu^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} \left[ q_i(\xi,\mu^2) P_{qq}\left(\frac{x}{\xi}\right) + g(\xi,\mu^2) P_{qg}\left(\frac{x}{\xi}\right) \right]$$

# Collinear factorization: DGLAP

- introduce new scale $\mu^2 \gg \chi^2$ and include soft, non-perturbative physics into renormalized parton density:

$$q_i(x,\mu^2) = q_i^0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[q_i^0(\xi)P_{qq}\left(\frac{x}{\xi}\right) + g^0(\xi)P_{qg}\left(\frac{x}{\xi}\right)\right]\log\left(\frac{\mu^2}{\chi^2}\right)$$

- DokshitzerGribovLipatovAltarelliParisi equation:

V.V. Gribov and L.N. Lipatov Sov. J. Nucl. Phys. 438 and 675 (1972) 15, L.N. Lipatov Sov. J. Nucl. Phys 94 (1975) 20, G. Altarelli and G. Parisi Nucl.Phys.B 298 (1977) 126, Y.L. Dokshitser Sov. Phys. JETP 641 (1977) 46

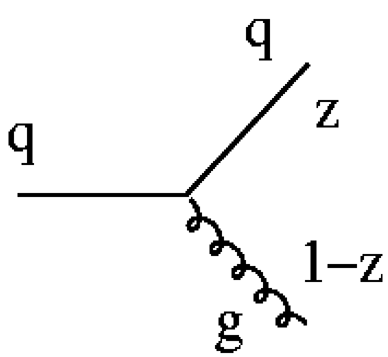
$$\frac{dq_i(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[q_i(\xi,\mu^2)P_{qq}\left(\frac{x}{\xi}\right) + g(\xi,\mu^2)P_{qg}\left(\frac{x}{\xi}\right)\right]$$

- BUT there are also gluons....

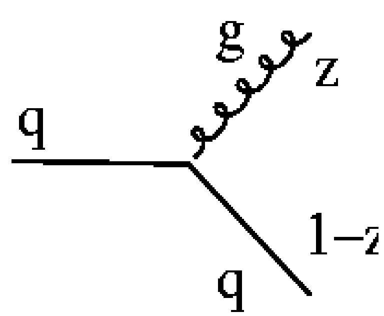
$$\frac{dg(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[\sum_i q_i(\xi,\mu^2)P_{gq}\left(\frac{x}{\xi}\right) + g(\xi,\mu^2)P_{gg}\left(\frac{x}{\xi}\right)\right]$$

- DGLAP is the analogue to the beta function for running of the coupling

# Splitting functions in lowest order

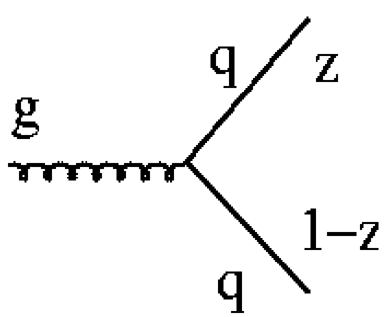



$$P_{qq} = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$$

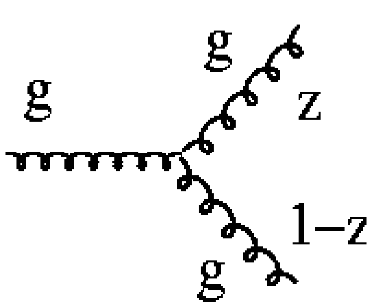



$$P_{gq} = \frac{4}{3}\left(\frac{1+(1-z)^2}{z}\right)$$

similarity to EPA...




$$P_{qg} = \frac{1}{2}\left(z^2+(1-z)^2\right)$$




$$P_{gg} = 6\left(\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right)$$

# Collinear factorization

see handbook of pQCD, chapter IV, B

$$F_2^{(Vh)}(x,Q^2) = \sum_{i=f,\bar{f},G} \int_0^1 d\xi\, C_2^{(Vi)}\left(\frac{x}{\xi},\frac{Q^2}{\mu^2},\frac{\mu_f^2}{\mu^2},\alpha_s(\mu^2)\right) \otimes f_{i/h}(\xi,\mu_f^2,\mu^2)$$

- Factorization Theorem in DIS (Collins, Soper, Sterman, (1989) in Pert. QCD, ed. A.H. Mueller, Wold Scientific, Singapore, p1.)
  - hard-scattering function $C_2^{(Vi)}$ is infrared finite and calculable in pQCD, depending only on vector boson *V*, parton *i*, and renormalization and factorization scales. It is independent of the identity of hadron *h*.
  - pdf $f_{i/h}(\xi,\mu_f^2,\mu^2)$ contains all the infrared sensitivity of cross section, and is specific to hadron h, and depends on factorization scale.
- Generalization: applies to any DIS cross section defined by a sum over hadronic final states .... but be careful what it really means....
- explicit factorization theorems exist for:
  - diffractive DIS (... see above....)
  - Drell Yan (in hadron hadron collisions)
  - single particle inclusive cross sections (fragmentation functions)

# Factorization proofs and all that ...

- About factorization proofs (Wu-Ki Tung, pQCD and the parton structure of the nucleon, 2001, In *Shifman, M. (ed.): At the frontier of particle physics, vol. 2* 887-971)

$$\frac{d\sigma}{dy} = \sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B f_A^a(\xi_A,\mu) f_B^b(\xi_B,\mu) \frac{d\hat{\sigma}_{ab}(\mu)}{dy} + \mathcal{O}\left(\left(\frac{m}{P}\right)^p\right)$$

- The problem with Drell-Yan: initial state interactions...
- factorization here does not hold graph-by-graph but only for all ....

# Collinear factorization ....

Ellis, Webber, Stirling, 123
Roberts 108

- So far considered only *"leading twist"*

  twist = dimension (spin) of operators in Operator Product Expansion (OPE)

- Factorization theorem (Collins hep-ph/9709499):

$$F_2(x, Q^2) = \sum_i C_{2\,i} \otimes f_i + \text{non-leading power of } Q$$

- in general:

$$F_2(x, Q^2) = \sum_n \frac{B_n(x, Q^2)}{Q^{2n}}$$

*n*>0 higher twists
non-leading powers ...

- NOT covered by factorization theorem.... but contributions can be large ?!?

# Warning on factorisation

# But even this is not the full story...

- factorization breaking in $pp \to j_1 j_2 X$

  J. Collins, J.W. Qiu hep-ph 0705.2141

- factorization breaking also in $tt$ production at large $p_t{}^{top}$

  S. Catani, M. Grazzini, and A. Torre. Transverse-momentum resummation for heavy-quark hadropro- duction. arXiv 1408.4564

# Collinear factorization schemes

- DIS scheme: absorbing all finite contributions $C_q$ into quark densities, with no finite $\mathcal{O}(\alpha_s)$ corrections:

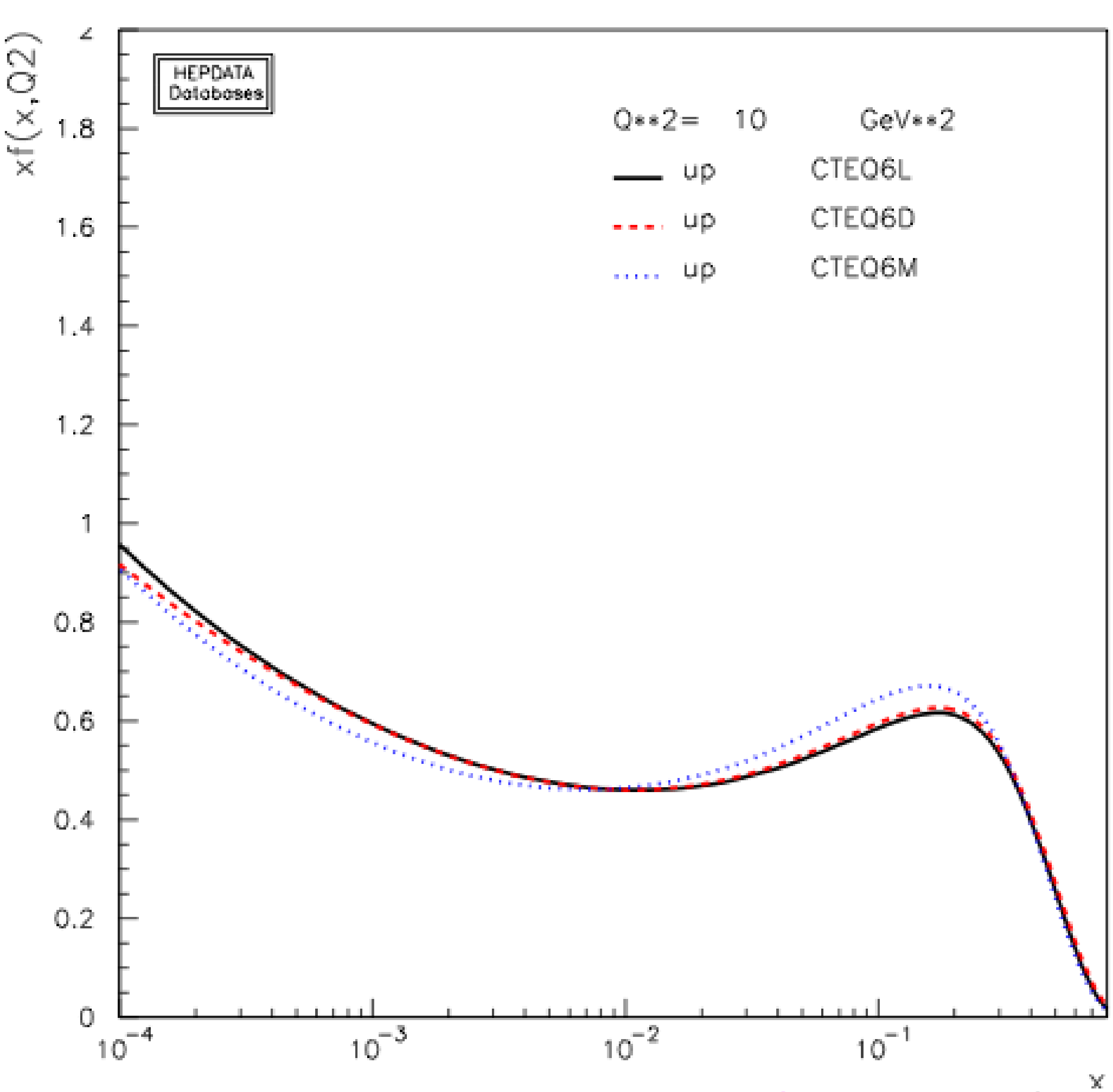
$$F_2^{DIS}(x, Q^2) = x \sum e_q^2 q(x, Q^2)$$

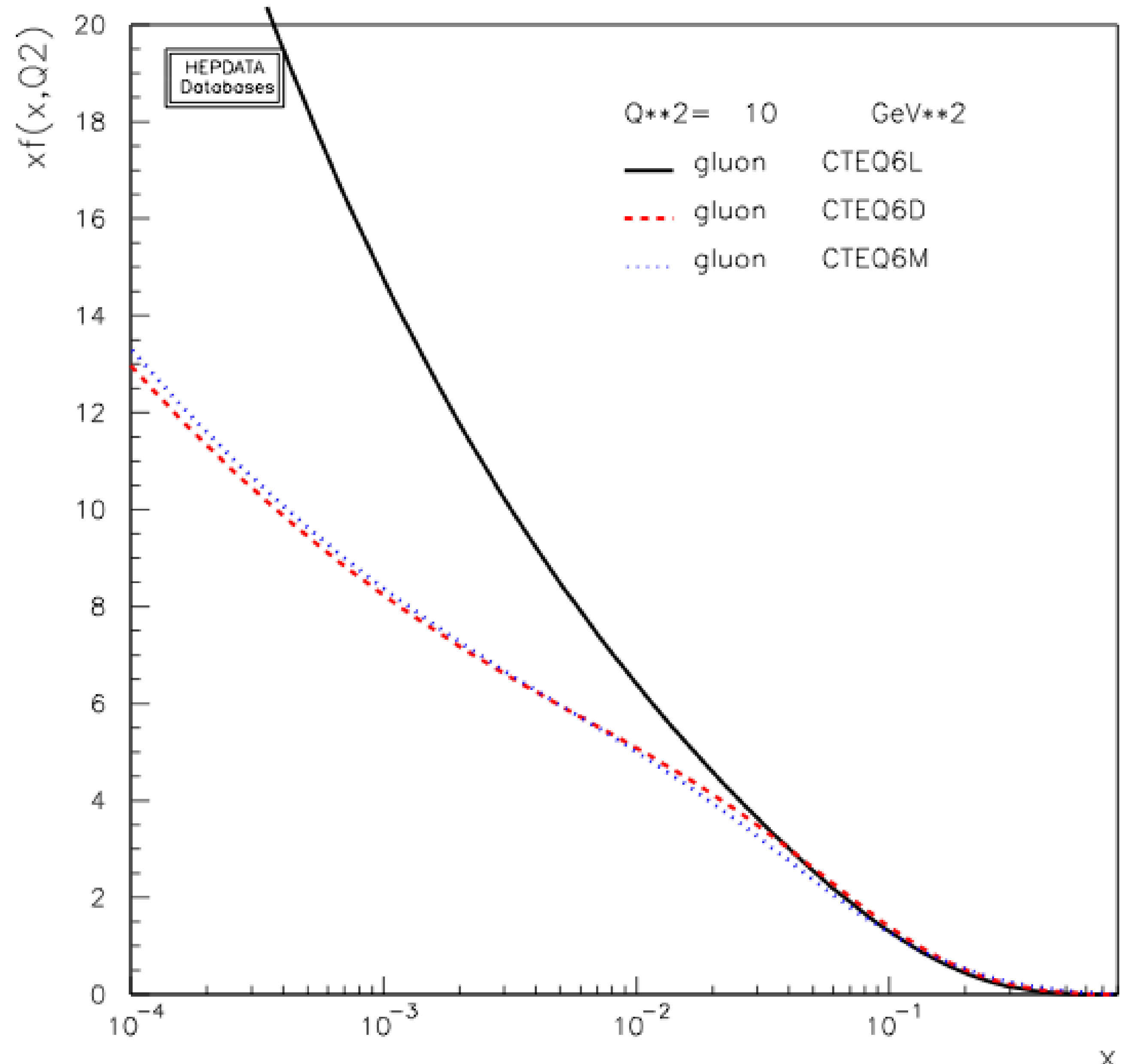
- $\overline{MS}$ scheme: only minimal contributions from the finite parts are absorbed in the quark distributions:

$$F_2^{\overline{MS}}(x, Q^2) = x \sum e_q^2 \int \frac{dx_2}{x_2} q^{\overline{MS}}(x, Q^2) \left[ \delta\left(1 - \frac{x}{x_2}\right) + \frac{\alpha_s}{2\pi} C^{\overline{MS}}\left(\frac{x}{x_2}\right) + ... \right]$$

- once the scheme is chosen, it MUST be used in all other cross section calculations
- higher order corrections will of course depend on the chosen scheme...
- **BUT....** there are still other contributions to be included... gluon induced processes

# PDFs in different fact. schemes




- differences between LO and NLO DIS, $\overline{MS}$ scheme in quark and gluon densities
- can make significant effects for x-sections

# But back to the evolution equation

# Splitting functions at higher orders

S. Moch, HERA-LHC workshop, June 2004

**The calculation (in a nut shell)**

$$P(z, \alpha_s) = P^{(0)}(z) + \frac{\alpha_s}{2\pi} P^{(1)}(z) + \ldots$$

– Calculate anomalous dimensions (Mellin moments of splitting functions)
$\longrightarrow$ divergence of Feynman diagrams in dimensional regularization $D = 4 - 2\varepsilon$

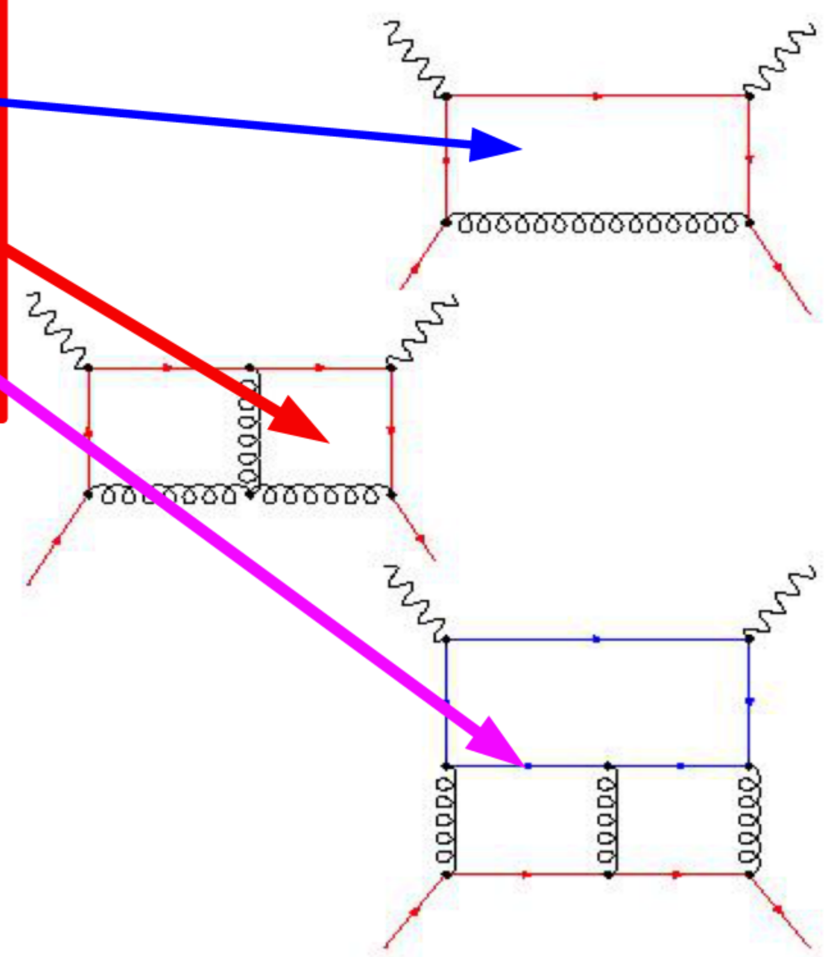
$$\gamma_{ij}^{(n)}(N) = -\int_0^1 dx\, x^{N-1} P_{ij}^{(n)}(x)$$

loops again:
1-loop
2-loops
3-loops

– **One-loop** Feynman diagrams
$\longrightarrow$ in total 18 for $\gamma_{ij}^{(0)}$ / $P_{ij}^{(0)}$
(pencil + paper)

– **Two-loop** Feynman diagrams
$\longrightarrow$ in total 350 for $\gamma_{ij}^{(1)}$ / $P_{ij}^{(1)}$
(simple computer algebra)

– **Three-loop** Feynman diagrams
$\longrightarrow$ in total 9607 for $\gamma_{ij}^{(2)}$ / $P_{ij}^{(2)}$
(cutting edge technology $\longrightarrow$ computer algebra system FORM Vermaseren '89-'04)



# Splitting functions (cont'd)

S. Moch, HERA-LHC workshop, June 2004

## LO and NLO singlet splitting functions

$$P(z,\alpha_s) = P^{(0)}(z) + \frac{\alpha_s}{2\pi} P^{(1)}(z) + \ldots$$

$$\begin{aligned}
P^{(0)}_{\rm ps}(x) &= 0 \\
P^{(0)}_{\rm qg}(x) &= 2 n_f\, p_{\rm qg}(x) \\
P^{(0)}_{\rm gq}(x) &= 2 C_F\, p_{\rm gq}(x) \\
P^{(0)}_{\rm gg}(x) &= C_A \Big(4 p_{\rm gg}(x) + \frac{11}{3}\delta(1-x)\Big) - \frac{2}{3} n_f\, \delta(1-x)
\end{aligned}$$

$$\begin{aligned}
P^{(1)}_{\rm ps}(x) &= 4 C_F n_f \Big(\frac{20}{9}\frac{1}{x} - 2 + 6x - 4{\rm H}_0 + x^2\Big[\frac{8}{3}{\rm H}_0 - \frac{56}{9}\Big] + (1+x)\Big[5{\rm H}_0 - 2{\rm H}_{0,0}\Big]\Big)
\end{aligned}$$

$$\begin{aligned}
P^{(1)}_{\rm qg}(x) &= 4 C_A n_f \Big(\frac{20}{9}\frac{1}{x} - 2 + 25x - 2p_{\rm qg}(-x){\rm H}_{-1,0} - 2p_{\rm qg}(x){\rm H}_{1,1} + x^2\Big[\frac{44}{3}{\rm H}_0 - \frac{218}{9}\Big] \\
&+4(1-x)\Big[{\rm H}_{0,0} - 2{\rm H}_0 + x{\rm H}_1\Big] - 4\zeta_2 x - 6{\rm H}_{0,0} + 9{\rm H}_0\Big) + 4 C_F n_f \Big(2p_{\rm qg}(x)\Big[{\rm H}_{1,0} + {\rm H}_{1,1} + {\rm H}_2 \\
&-\zeta_2\Big] + 4x^2\Big[{\rm H}_0 + {\rm H}_{0,0} + \frac{5}{2}\Big] + 2(1-x)\Big[{\rm H}_0 + {\rm H}_{0,0} - 2x{\rm H}_1 + \frac{29}{4}\Big] - \frac{15}{2} - {\rm H}_{0,0} - \frac{1}{2}{\rm H}_0\Big)
\end{aligned}$$

$$\begin{aligned}
P^{(1)}_{\rm gq}(x) &= 4 C_A C_F \Big(\frac{1}{x} + 2p_{\rm gq}(x)\Big[{\rm H}_{1,0} + {\rm H}_{1,1} + {\rm H}_2 - \frac{11}{6}{\rm H}_1\Big] - x^2\Big[\frac{8}{3}{\rm H}_0 - \frac{44}{9}\Big] + 4\zeta_2 - 2 \\
&-7{\rm H}_0 + 2{\rm H}_{0,0} - 2{\rm H}_1 x + (1+x)\Big[2{\rm H}_{0,0} - 5{\rm H}_0 + \frac{37}{9}\Big] - 2p_{\rm gq}(-x){\rm H}_{-1,0}\Big) - 4 C_F n_f \Big(\frac{2}{3}x \\
&-p_{\rm gq}(x)\Big[\frac{2}{3}{\rm H}_1 - \frac{10}{9}\Big]\Big) + 4 C_F^2 \Big(p_{\rm gq}(x)\Big[3{\rm H}_1 - 2{\rm H}_{1,1}\Big] + (1+x)\Big[{\rm H}_{0,0} - \frac{7}{2} + \frac{7}{2}{\rm H}_0\Big] - 3{\rm H}_{0,0} \\
&+1 - \frac{3}{2}{\rm H}_0 + 2{\rm H}_1 x\Big)
\end{aligned}$$

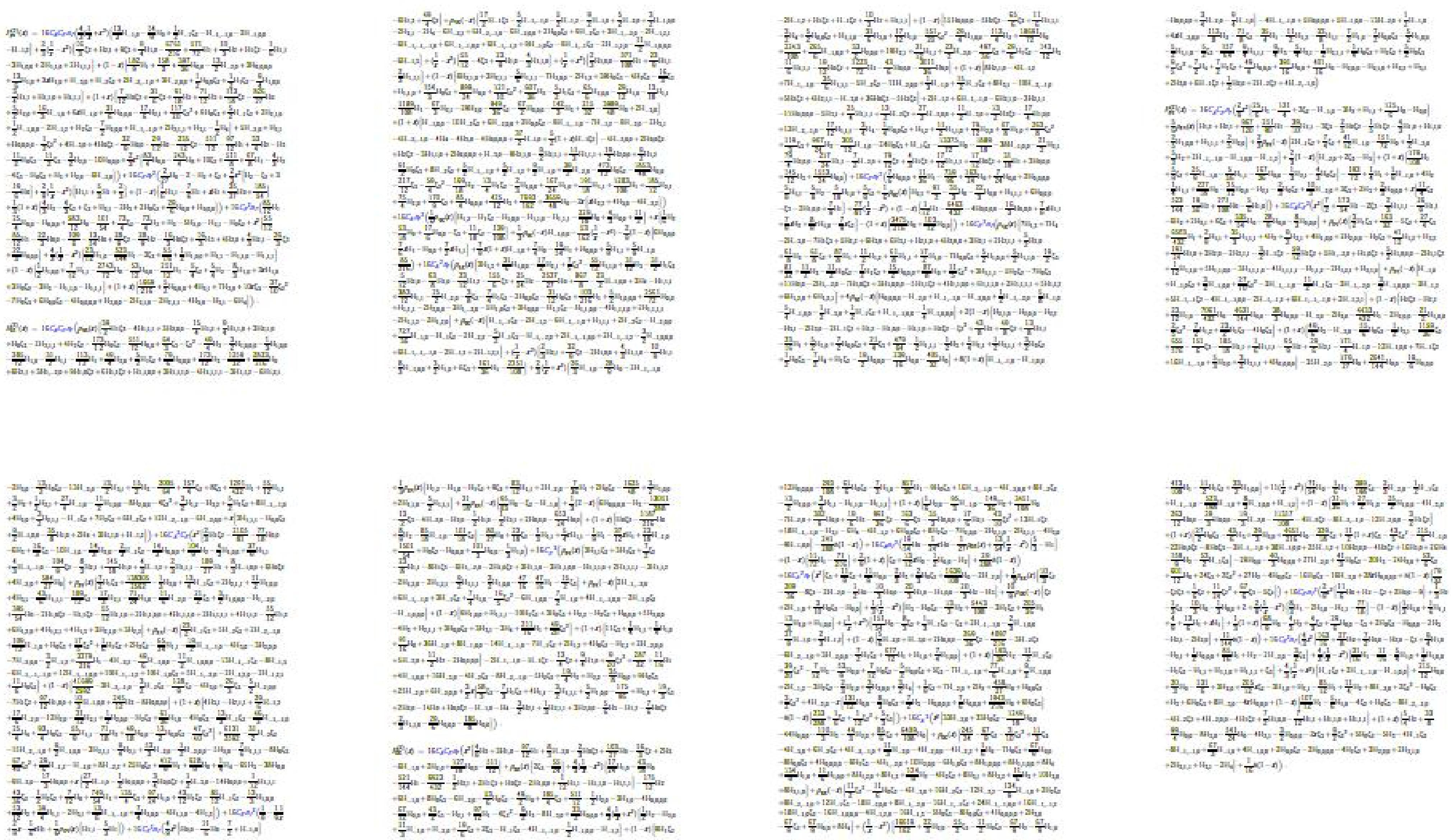
$$\begin{aligned}
P^{(1)}_{\rm gg}(x) &= 4 C_A n_f \Big(1 - x - \frac{10}{9}p_{\rm gg}(x) - \frac{13}{9}\Big(\frac{1}{x} - x^2\Big) - \frac{2}{3}(1+x){\rm H}_0 - \frac{2}{3}\delta(1-x)\Big) + 4 C_A^2 \Big(27 \\
&+(1+x)\Big[\frac{11}{3}{\rm H}_0 + 8{\rm H}_{0,0} - \frac{27}{2}\Big] + 2p_{\rm gg}(-x)\Big[{\rm H}_{0,0} - 2{\rm H}_{-1,0} - \zeta_2\Big] - \frac{67}{9}\Big(\frac{1}{x} - x^2\Big) - 12{\rm H}_0 \\
&-\frac{44}{3}x^2{\rm H}_0 + 2p_{\rm gg}(x)\Big[\frac{67}{18} - \zeta_2 + {\rm H}_{0,0} + 2{\rm H}_{1,0} + 2{\rm H}_2\Big] + \delta(1-x)\Big[\frac{8}{3} + 3\zeta_3\Big]\Big) + 4 C_F n_f \Big(2{\rm H}_0 \\
&+\frac{2}{3}\frac{1}{x} + \frac{10}{3}x^2 - 12 + (1+x)\Big[4 - 5{\rm H}_0 - 2{\rm H}_{0,0}\Big] - \frac{1}{2}\delta(1-x)\Big)\,.
\end{aligned}$$

# Splitting functions (cont'd)

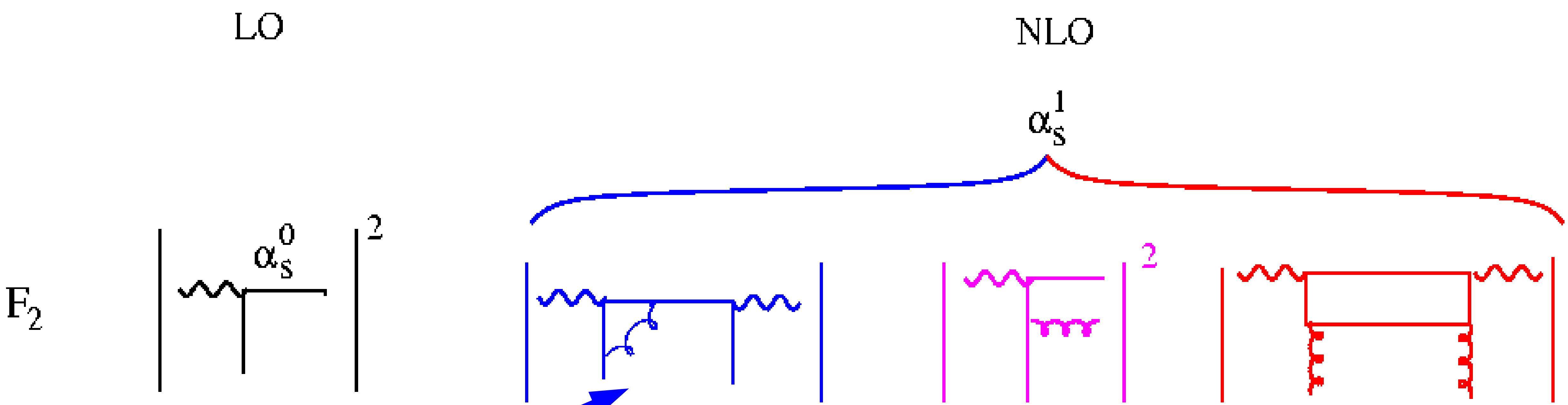
$$P(z, \alpha_s) = P^{(0)}(z) + \frac{\alpha_s}{2\pi} P^{(1)}(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 P^{(2)}(z) + \ldots$$

S. Moch, HERA-LHC workshop, June 2004

**NNLO singlet splitting functions**

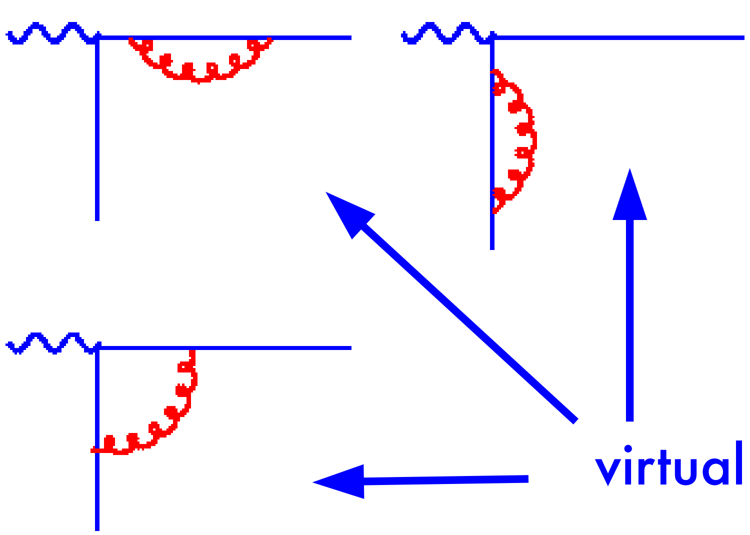


# NLO contributions to $F_2(x,Q^2)$




virtual corrections cancel collinear singularity

$$P_{qq} = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$$



virtual corrections to QPM

$$P_{qg} = \frac{1}{2}\left(z^2 + (1+z)^2\right)$$

no collinear singularity

# Evolution kernels – splitting fcts

- flavor conservation

$$
\begin{aligned}
q(x,\mu^2) &= \int_x^1 \frac{d\xi}{\xi} q_0(\xi)\left[\delta(1-\frac{x}{\xi}) + \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right)\log\frac{\mu^2}{\chi^2} + \ldots\right] \\
&= \int_x^1 \frac{d\xi}{\xi} q_0(\xi)\hat{q}(z,\mu^2) + \ldots \\
&= \int_x^1 d\xi \int_0^1 dz \delta(x - z\xi) q_0(\xi)\hat{q}(z,\mu^2) + \ldots
\end{aligned}
$$

$$
\hat{q}(z,\mu^2) = \delta(1-z) + \frac{\alpha_s}{2\pi} P_{qq}(z)\log\frac{\mu^2}{\chi^2} \qquad \int_0^1 dz\, \hat{q}(z,\mu^2) = 1
$$

with

$$
P_{qq}(z) = \hat{P}_{qq}(z) + k\cdot\delta(1-z)
$$

and

$$
\int dz \left[\delta(1-z) + \frac{\alpha_s}{2\pi}\left(\hat{P}_{qq}(z) + k\cdot\delta(1-z)\right)\log\frac{\mu^2}{\chi^2}\right] = 1
$$

redefine splitting function:

obtain:

$$
\int_0^1 dz \frac{\alpha_s}{2\pi}\left(\hat{P}(z) + k\cdot\delta(1-z)\right) = 0
$$

# Evolution kernels – splitting fcts

- using $$\int_0^1 dx \frac{f(x)}{(1-x)_+} = \int_0^1 dx \frac{f(x)-f(1)}{(1-x)}$$

we can calculate for the quark sector with $\hat{P}_{qq}(z) = \frac{1+z^2}{(1-z)_+}$ :

$$\begin{aligned}
\int_0^1 dz P_{qq}(z) &= \int_0^1 dz \left[\frac{1+z^2}{(1-z)_+} + k\cdot\delta(1-z)\right] \\
&= \int_0^1 dz \, \frac{1+z^2-2}{1-z} + k \\
&= k + \int_0^1 dz \frac{-(1-z^2)}{1-z} \\
&= k - \int_0^1 dz \frac{(1+z)(1-z)}{1-z} \\
&= k - \int_0^1 dz(1+z) = k - \frac{3}{2}
\end{aligned}$$

$$P_{qq}(z) = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)$$

# Evolution kernels – splitting fcts

- some of the splitting functions are also divergent...

$$\frac{1}{1-z}$$

- use *plus-distribution* to avoid dangerous region:

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z)-f(1)}{1-z}$$

- divergence cancelled by virtual corrections ...
- use splitting functions with plus-distribution

# Conservation rules with DGLAP

$$\int_0^1 dx x \left[ \sum_{i=-6}^{6} q(x,\mu^2) + g(x,\mu^2) \right] = 1$$

- use DGLAP

$$\frac{dq_i(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[ q_i(\xi,\mu^2) P_{qq}\left(\frac{x}{\xi}\right) + g(\xi,\mu^2) P_{qg}\left(\frac{x}{\xi}\right)\right]$$

$$\frac{dg(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[ \sum_i q_i(\xi,\mu^2) P_{gq}\left(\frac{x}{\xi}\right) + g(\xi,\mu^2) P_{gg}\left(\frac{x}{\xi}\right)\right]$$

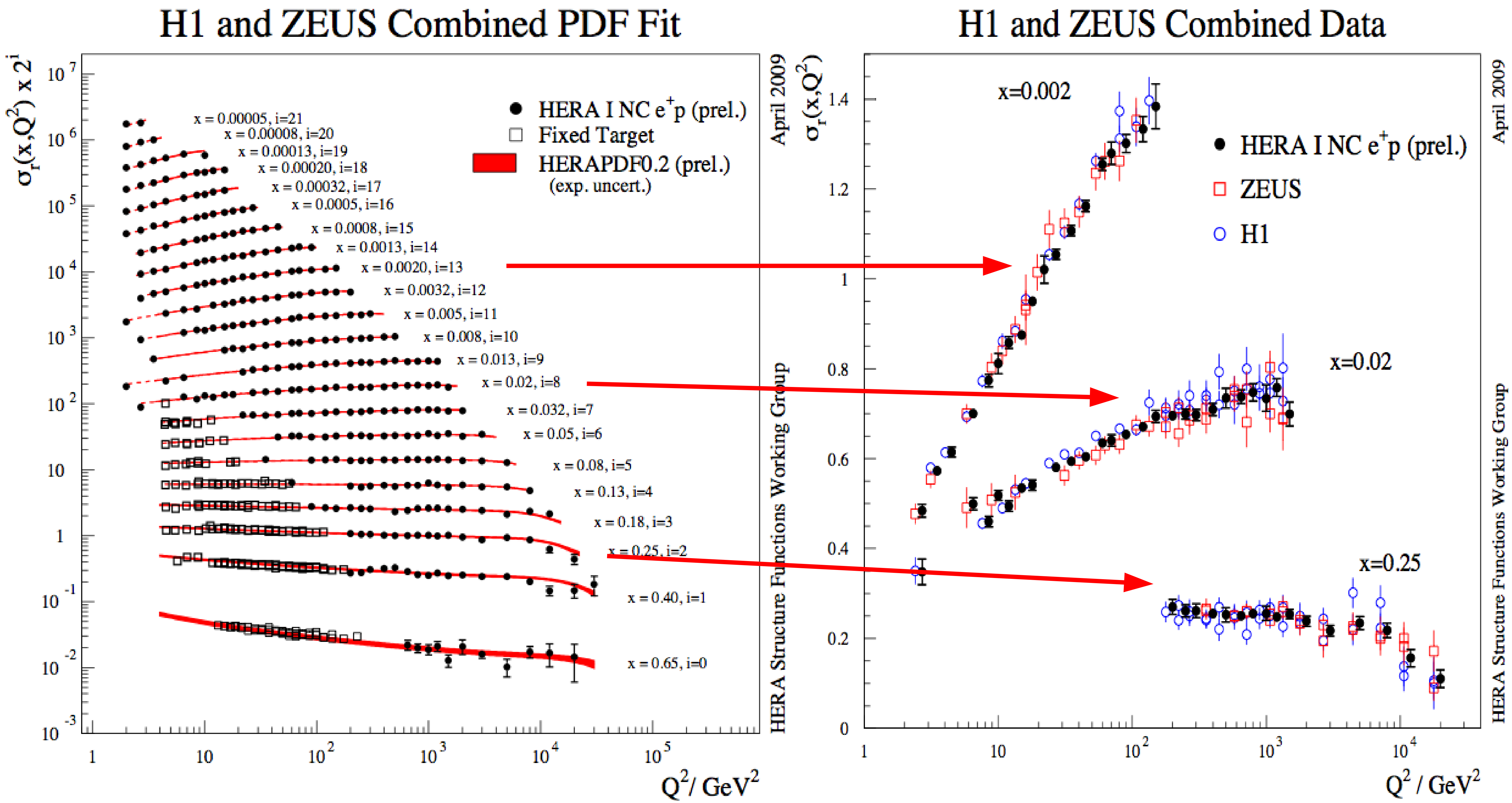
➔to obtain:

$$\int_0^1 dx\, x \left[P_{qq}(x) + P_{gq}(x)\right] = 0$$

$$\int_0^1 dx\, x \left[P_{gg}(x) + 2n_f P_{qg}(x)\right] = 0$$

# How to apply these results

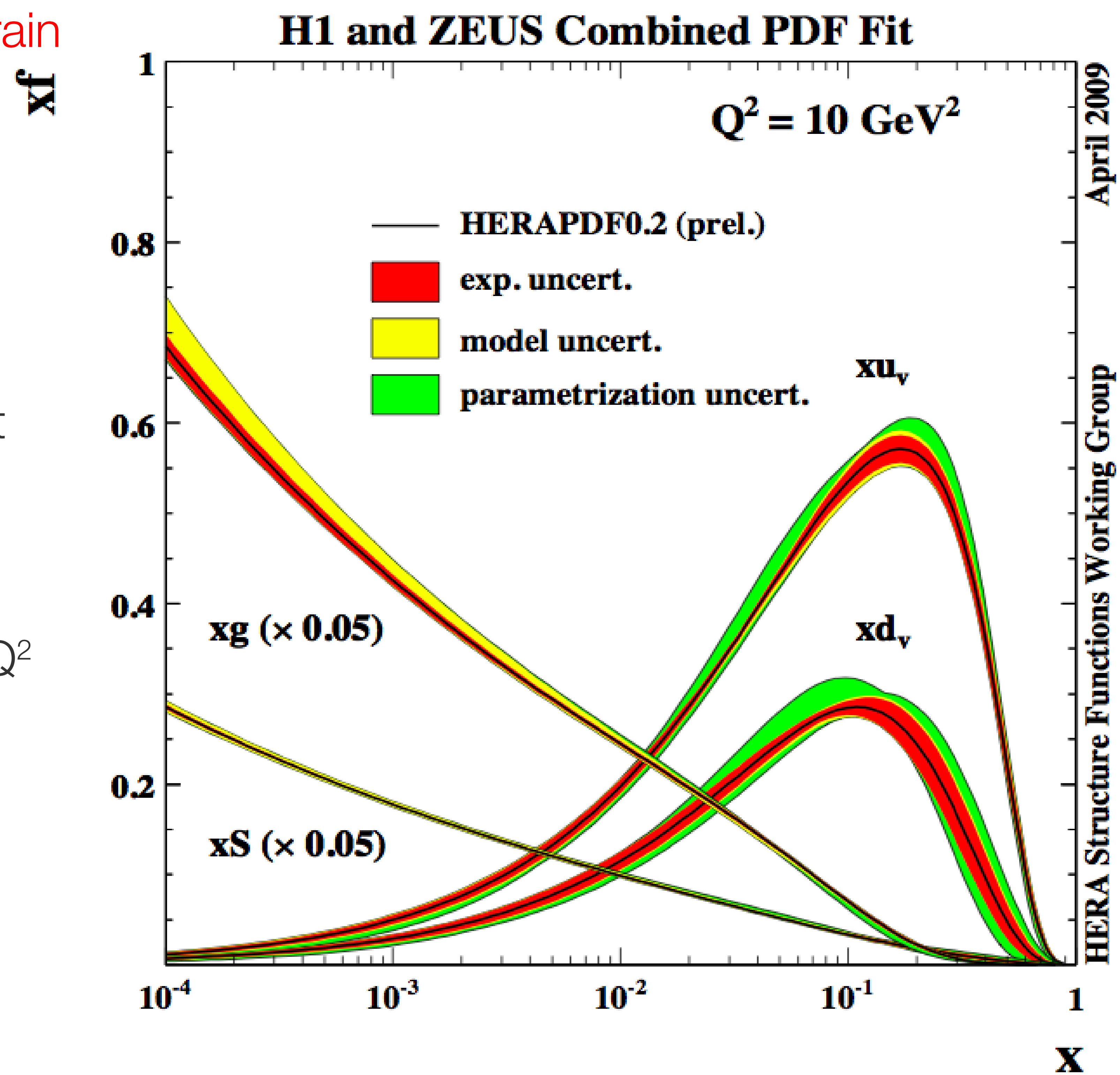
# Applying DGLAP to DIS data ...




- Theory describes measurement over huge range in x and $Q^2$
- Success of theory (DGLAP)

# Extraction of PDFs from DGLAP fits

- Sum rules are essential to constrain starting distributions
- Solve DGLAP equations
- adjust input parameters (starting distributions) such that F2 is best described
- extract PDFs as fct of x
- then DGLAP gives PDFs at any $Q^2$



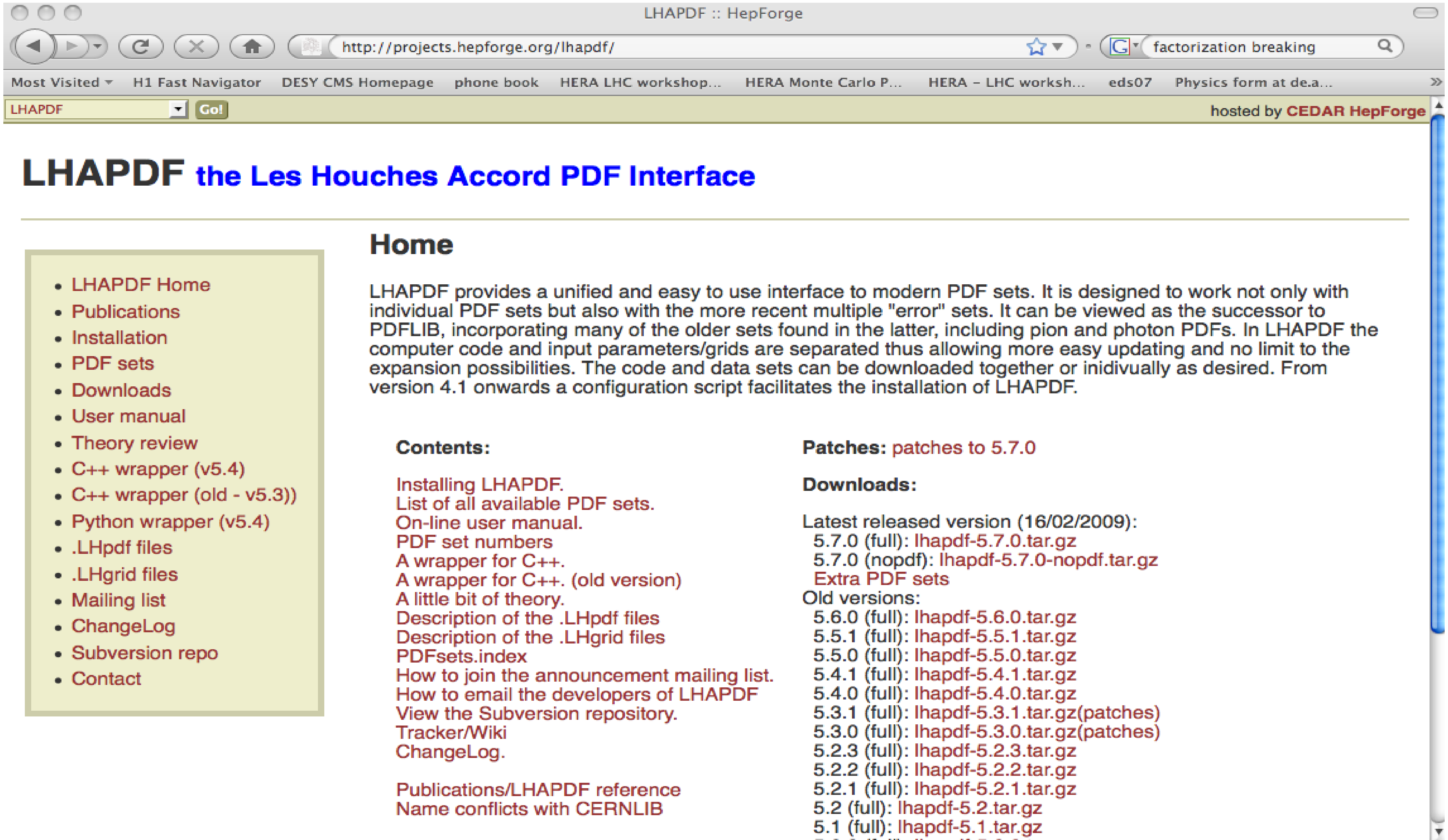
# Solving DGLAP equations ...

- Different methods to solve integro-differential equations
  - brute-force (BF) method (M. Miyama, S. Kumano CPC 94 (1996) 185)

$$\frac{df(x)}{dx} = \frac{f(x)_{m+1} - f(x)_m}{\Delta x_m} \qquad \int f(x)dx = \sum f(x)_m \Delta x_m$$

  - Laguerre method (S. Kumano J.T. Londergan CPC 69 (1992) 373, and L. Schoeffel Nucl.Instrum.Meth.A423:439-445,1999)
  - Mellin transforms (M. Glueck, E. Reya, A. Vogt Z. Phys. C48 (1990) 471)
  - QCDNUM: calculation in a grid in x,Q2 space (M. Botje Eur.Phys.J. C14 (2000) 285-297)
  - CTEQ evolution program in x,Q2 space: http://www.phys.psu.edu/~cteq/
  - QCDFIT program in x,Q2 space (C. Pascaud, F. Zomer, LAL preprint LAL/94-02, H1-09/94-404,H1-09/94-376)
  - MC method using Markov chains (S. Jadach, M. Skrzypek hep-ph/0504205)
  - Monte Carlo method from iterative procedure
- brute-force method and MC method are best suited for detailed studies of branching processes !!!

# Evolution code in LHAPDF




Can use LHAPDF to evolve starting distribution to any $Q^2$ with

- CTEQ, QCDNUM, and other evolution packages...