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QCD and Monte Carlo method

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Contents

Chapter 1

Introduction

This writeup is based on a series of lectures on *QCD and Collider physics* and *QCD and MC* given in the years 2005 – 2015 at University Hamburg/DESY, at University of Antwerp and at IFJ PAN Cracow. The full writeup, lecture notes, exercises and solutions can be found at `https://www.desy.de/ jung/QCD_and_Monte_Carlo_lectures.html`, where also more details and calculations are given.
The aim is to introduce the basic concepts of QCD and how this can be used for comparison with measurements at the past and present high energy particle colliders, HERA and the LHC. Since events produced at high energy collisions contain many particles, most of the calculations cannot be performed analytically. Even for the calculation of integrals, numerical methods have to be applied and for complicated multidimensional integrals the Monte Carlo method is best suited. The basics of the MC method will be discussed in chapter 2. The basics of QCD and the naive quark parton model will be discussed in Chapter 2, with its extension to include QCD effects, where the parton evolution equations will be discussed.
While the basic concepts and methods did not change in the last few years, the experimental results and the interest to understand the measurements has changed: LHC has started and the experiments have published already within one year a large number of measurements, many of them confirming the predictions coming from QCD or more generally from the Standard Model of Particle Physics, but also some which came as a big surprise.
A warning is needed here: although the lecture will cover Monte Carlo methods, it will not be a description how to run a given Monte Carlo event generator, nor it will describe the detailed implementation of QCD processes in Monte Carlo generators. The lectures will provide the basics to understand the principles of Monte Carlo event simulation and basic QCD calculations.

Chapter 2

Monte Carlo methods

The general case of a process $A + B \rightarrow$ anything to be calculated is given in fig. 2.1. A more

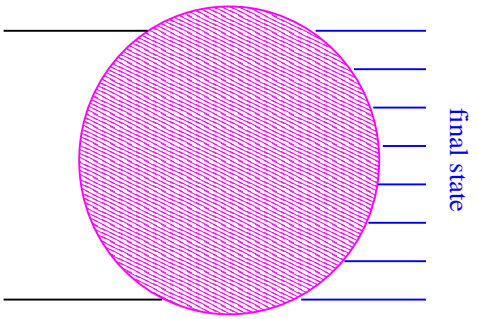

Figure 2.1: General case of scattering $A + B \rightarrow$ anything

detailed figure of the process to be studied is shown in fig 2.2, where on the left side is shown the lowest order process for jet production in hadron hadron collisions and on the right side the process is shown including multiparton radiation, which is the subject of this lecture. It becomes clear, that with many partons[1] involved in the calculation this cannot be done analytically, and numerical methods are needed, one of them is the Monte Carlo method.

2.1 Random Numbers

Monte Carlo method refers to any procedure, which makes use of random numbers and uses probability statistics to solve the problem[2]. The Monte Carlo method was invented and

[1] Partons are used as a generic name for quarks and gluons

[2] The name comes from a saga, that the first true random numbers were obtained by recording the results of the roulette game in the casino of Monte Carlo.

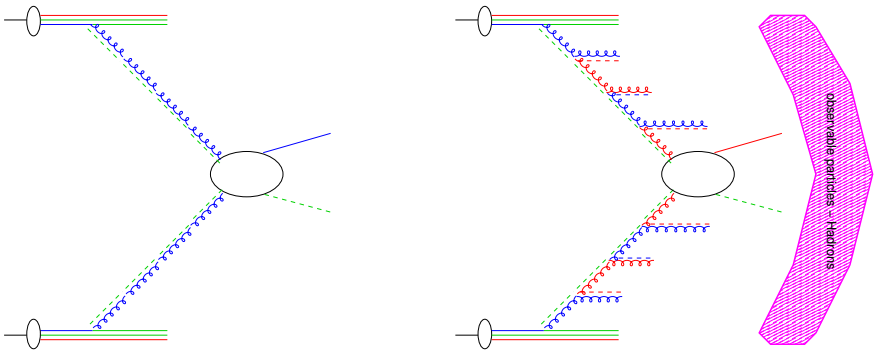

Figure 2.2: Left: lowest order process for jet production in hadron hadron collisions. Right: Process for jet production including multiparton radiation and hadronization

developed in the 1930's for the calculation of nuclear decays, but nowadays is widely used in any calculation of complicated processes for the simulation of natural phenomena, simulation of the experimental apparatus, simulation of the underlying physics process but also in economy for risk analysis etc.
Monte Carlo methods make use of random numbers. An example of a random number is 3 or 4. There is nothing like *a random number*. Any number can appear to be random. Only if we have a sequence of numbers, where each number has nothing to do with the other numbers in the series, we can say the numbers appear to be random.
In the following we consider random numbers always only in the interval $[0, 1]$. In a uniform distribution of random numbers in this interval $[0, 1]$ all numbers have the same chance to appear, note that 0.00000034 has the same chance to appear as 0.5.
Random numbers can be obtained by several methods:

- using a truly chaotic system like roulette, lotto or 6-49
- using a process which is inherently random
- generating "random numbers" on a computer

Examples for random numbers obtained from chaotic processes are using atmospheric noise [1] or using quantum physics which is intrinsically random [2].
Random numbers generated on a computer are never really random, since they always are determined according to some algorithm [3]. They may appear random to someone, who does not know the algorithm. The randomness of random numbers can be checked by several test, which will be discussed later. Random numbers, which are generated on a computer are called *pseudo-random numbers*. Sometime *quasi-random numbers* are discussed. Such random numbers are by intention not random but are designed to be as uniform as possible in order to minimize the uncertainties in integration procedures.

A simple random number generator (so called *multiplicative congruential linear random number generator*) can be build as follows [4][p 40ff] and [5][Vol II,p 9]. From an initial number I_0 we generate a series of random numbers R_j according to:

$$\begin{aligned} I_j &= mod(aI_{j-1} + c, m) \\ R_j &= \frac{I_j}{m} \end{aligned} \tag{2.1}$$

with a being an multiplicative and c a additive constant and m the modulus[3]. With this procedure one obtains a series of number R_j in the interval $(0, 1)$ (note that the values 0 and 1 are excluded). This random number generator will be tested in the exercise. In fig 2.3 the correlation of 2 random numbers is shown on the left side. The right side shows the same correlation for another random number generator RANLUX [6–8], which will be used later in the calculations. It is obvious, that the *multiplicative congruential linear random number generator* produces random numbers, which show correlations and does therefore not satisfy quality criteria for a good random number generator; the RANLUX generator seems to be better.

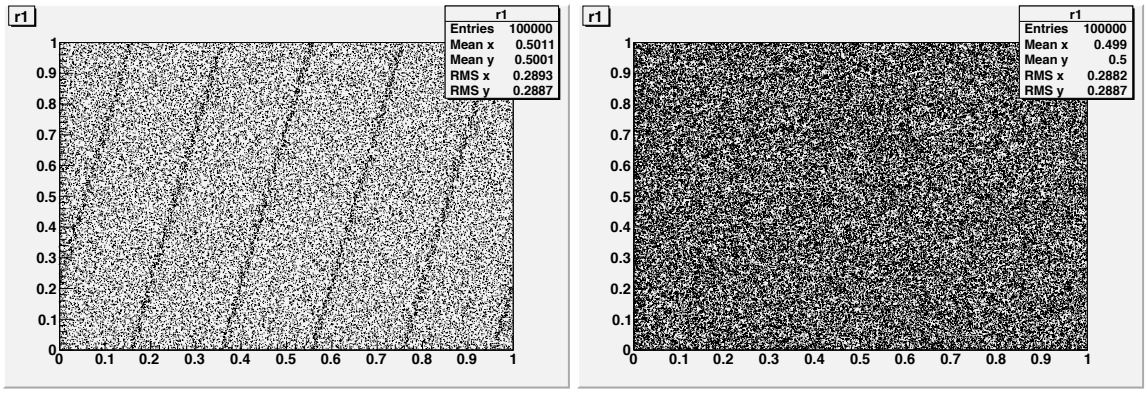

Figure 2.3: Left: correlation of two successive random numbers obtained according to 2.1. Right: correlation of two random numbers obtained with RANLUX [8]

Several criteria on the randomness of a series of *pseudo random numbers* can be applied to test the quality of the random number generator [5][Vol II,p 59]:

- **statistical test** (test uniformity of distribution, frequency test, equi-distribution test) Divide the interval $(0, 1)$ into k-subintervals with length $1/k$. Count how many random numbers fall into the k's interval [9]. Calculate:

$$\chi^2 = \sum_{i=1}^{k} \frac{(N_i - N/k)^2}{N/k} \tag{2.2}$$

[3]the modulus function is defined as $mod(i_1, i_2) = i_1 - INT(i_1/i_2) \cdot i_2$

with N random numbers R_i. If the random numbers are uniformly distributed, then Eq. 2.2 is a χ^2 distribution with $k-1$ degrees of freedom and should give $\chi^2/(ndf) \sim 1$, with ndf being the number of degrees of freedom.

- **serial test** (pairs of successive random numbers should be distributed in an independent way (see fig. 2.3)). *The sun comes up just about as often as it goes down, in the long run, but this does not make its motion random* [5][Vol II,p 60].
  Count pairs of random numbers $(Y_{2j}, Y_{2j+1}) = (q, r)$ for any $0 \leq j \leq n$ and apply a χ^2 test as above.

- **sequence up-down test**
  Count the number of runs, where the random numbers are increasing $Y_{j+1} > Y_j$. Example: take the sequence 1298536704 and insert vertical lines for $Y_{j+1} > Y_j$, resulting in $|129|8|5|367|0|4|$. Count the number of runs-up with length k. The number of runs-up and the number of runs-down should be similar, but they should not be adjacent: often a long run will be followed by a short one.

- **gap test**
  Choose two numbers α, β with $0 \leq \alpha < \beta \leq 1$. Generate $r+1$ random numbers. The probability that the first r random numbers are outside (α, β) is $P_r = p(1-p)^r$ with $p = \beta - \alpha$ being the probability for the $r+1$ event to be inside (α, β).

- **Random walk test**
  Choose $0 \leq \alpha \leq 1$ and generate a large number of random variables. Count how often $Y_i < \alpha$ and call it r. We expect a binominal distribution for r with $p = \alpha$. The same test can be performed for $Y_i > (1-\alpha)$.

Practical criteria for random numbers can be formulated as follows [3]:

- **Long period**

- **Repeatability**
  for testing and development one needs to repeat calculations. Repeatability also allows to repeat only part of the job, without re-doing the whole.

- **Long disjoint sequences**
  for long procedures one needs to be able to perform independent sub-calculations which can be added later.

- **Portability**
  not only the code should be portable but also the results should be the same, independent on which platform the calculations are done.

- **Efficiency**
  generation of random numbers should be fast.

To test a random number generator, a series of tests have to be performed. Even if a Random Number generator passes all n -tests, one cannot assume that it also passes the $n+1$-test.

2.2 Statistics and Probabilities

A very good overview on statistics and probabilities is given in [4, 10], which was used for the discussion in this chapter. In an experiment where the outcome depends on a single variable x one can ask what is the probability to find values of x in the interval $[x, x + dx]$. This is given by $f(x)dx$ with $f(x)$ being the probability density function *p.d.f* (not to be confused with the pdf which is used for the parton density function to be discussed later). Since we assume, there is an experiment with some outcome, the probability to find x anywhere must be unity, that is:

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{2.3}$$

The *p.d.f* has to satisfy in addition:

$$f(\infty) = f(-\infty) = 0 \tag{2.4}$$

The expectation value (mean values or average value) of a function $h(x)$ is defined as:

$$E[h] = \int_{-\infty}^{+\infty} h(x)f(x)dx = \int h(x)dG(x) = \frac{1}{b-a}\int h(x)dx \tag{2.5}$$

with $f(x)$ being the probability density function. In the right part of the equation we used the special case $dG(x) = dx/(b-a)$ for a uniform distribution. In case of discrete distributions we have:

$$E[h] = \sum_{i}^{\infty} h(x_i)f(x_i) \tag{2.6}$$

A special case is the expectation value of x (or the mean on the distribution)

$$E[x] = \int_{-\infty}^{+\infty} f(x)xdx \stackrel{def}{=} \langle x \rangle \tag{2.7}$$

From the definition of the expectation value we see that $E[h]$ is a **linear operator**:

$$\begin{aligned} E[cg(x) + h(x)] &= \int (cg(x) + h(x))\, f(x)dx \\ &= c\int g(x)dx + \int h(x)f(x)dx \\ &= cE[g] + E[h] \end{aligned} \tag{2.8}$$

with c being a constant. Similarly we can see that the expectation value of the expectation value $E[E[g]]$ is simply $E[g]$:

$$\begin{aligned} E\left[E[g(x)]\right] &= \int \left(\int g(x)f(x)dx\right) f(x')dx' \\ &= E[g(x)]\int f(x')dx' \\ &= E[g(x)] \end{aligned} \tag{2.9}$$

because $\int f(x')dx' = 1$ by definition of the *p.d.f*
The variance σ^2 measures the spread of a distribution and can be defined as the mean quadratic deviation from the mean value. The square-root of σ^2 is also called the *standard deviation*. The variance $V[h]$ is defined as:

$$V[h] = \sigma^2 = E\left[(h(x) - E[h(x)])^2\right] = \int (h(x) - E[h(x)])^2 f(x)dx \tag{2.10}$$

From the definition, the variance $V[cg(x) + h(x)]$ can be calculated:

$$\begin{aligned}
V[cg(x)+h(x)] &= \int (cg(x) + h(x) - E[cg(x) + h(x)])^2 f(x)dx \\
&= \int (cg(x) + h(x) - cE[g(x)] - E[h(x)])^2 f(x)dx \\
&= \int ((c(g - E[g]) + (h - E[h]))^2 f(x)dx \\
&= \int \left(c^2(g - E[g])^2 + 2c(g - E[g])(h - E[h]) + (h - E[h])^2\right) f(x)dx \\
&= c^2V[g] + 2cE[(g - E[g])(h - E[h])] + V[h] \\
&= c^2V[g] + V[h] + 2cE[g \cdot h - gE[h] - hE[g] + E[g]E[h]] \\
&= c^2V[g] + V[h] + 2c\left(E[g \cdot h] - E[g]E[h] - E[h]E[g] + E[g]E[h]\right) \\
V[cg(x)+h(x)] &= c^2V[g] + V[h] + 2c\left(E[g \cdot h] - E[g]E[h]\right)
\end{aligned} \tag{2.11}$$

In the case that $g(x)$ and $h(x)$ are uncorrelated, we have $E[g \cdot h] = E[g]E[h]$ and the expression simplifies to:

$$V[cg(x) + h(x)] = c^2V[g] + V[h] \tag{2.12}$$

A special case is the variance of x:

$$\begin{aligned}
V[x] &= E\left((x - \langle x\rangle)^2\right) = \int (x - E[x])^2 f(x)dx \\
&= E\left[x^2 - 2x\langle x\rangle + \langle x\rangle^2\right] \\
&= E[x^2] - 2E[x]\langle x\rangle + \langle x\rangle^2 \\
V[x] &= E[x^2] - \langle x\rangle^2 \\
V[x] &= E[x^2] - E[x]^2
\end{aligned} \tag{2.13}$$

where the relation $E[x] = \langle x\rangle$ has been applied.
Consider independent random numbers x_1 and x_2 with variances $V[x_1] = \sigma_1^2$ and $V[x_2] = \sigma_2^2$ and mean values μ_1 and μ_2. The expectation value of the sum of x_1 and x_2 is:

$$\begin{aligned}
E[x_1 + x_2] &= E[x_1] + E[x_2] \\
&= \mu_1 + \mu_2
\end{aligned} \tag{2.14}$$

The variance of the sum is (using $x = x_1 + x_2$):

$$\begin{aligned}
\sigma^2 &= \langle x - \langle x \rangle \rangle \\
&= E[(x - \langle x \rangle)^2] \\
&= E[(x - \mu_1 - \mu_2)^2] \\
&= E[(x - \mu_1 + x_2 - \mu_2)^2] \\
&= \underbrace{E[(x_1 - \mu_1)^2}_{\sigma_1^2} + 2\underbrace{(x_1 - \mu_1)(x_2 - \mu_2)}_{0} + \underbrace{(x_2 - \mu_2)^2]}_{\sigma_2^2} \\
\sigma^2 &= \sigma_1^2 + \sigma_2^2
\end{aligned} \tag{2.15}$$

because $E[x_1] = \mu_1$, since x_1 and x_2 are independent. The general form is then:

$$\sigma^2 = \sum_{i=1}^{N} \sigma_i^2 \tag{2.16}$$

Consider now a sample of x_i where all x_i follow the same probability density function $f(x)$, having the same variance σ^2 and the same μ. The mean of the sample is defined as:

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{2.17}$$

The expectation value $E[\bar{x}]$ is given by:

$$\begin{aligned}
E[\bar{x}] &= E\left[\frac{1}{N} \sum_{i=1}^{N} x_i\right] \\
&= \frac{1}{N} E\left[\sum_{i=1}^{N} x_i\right] \\
&= \frac{1}{N} N E[x_i] \\
E[\bar{x}] &= E[x] = \langle x \rangle
\end{aligned} \tag{2.18}$$

resulting in the expectation value of the mean being the mean itself. To obtain above, the features of the linearity of the operator are applied.
The variance of the mean is:

$$\begin{aligned}
V[\bar{x}] &= V\left[\frac{1}{N} \sum_{i=1}^{N} x_i\right] \\
&= \frac{1}{N^2} V\left[\sum x_i\right]
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{N^2}\sum \sigma_i^2 \\ &= \frac{1}{N^2} N\sigma^2 \\ V[\bar{x}] &= \frac{1}{N}\sigma^2 \end{aligned} \tag{2.19}$$

or in the familiar form as the standard deviation of the mean:

$$\sigma_N = \frac{\sigma}{\sqrt{N}} \tag{2.20}$$

2.3 Random Numbers from arbitrary distributions

Given a sequence of random numbers uniformly distributed in $[0, 1]$ the next step is to determine a sequence of random numbers $x_1, x_2 \ldots$ distributed according to a probability density function *p.d.f.*
The task is to find a suitable function $x(r)$ which gives the same sequence of random numbers when evaluated with uniformly distributed values r. The probability to obtain a value r in the interval $[r, r + dr]$ is $u(r)dr$ and this should be equal to the probability to find x in $[x, x + dx]$ which is $f(x)dx$ (see fig 2.4):

$$\begin{aligned} u(r')dr' &= f(x')dx' \\ \int_{-\infty}^{r} u(r')dr' &= \int_{-\infty}^{x} f(x')dx' \end{aligned} \tag{2.21}$$

Using a random number R uniform in $[0, 1]$ with $R = \int_{-\infty}^{r} u(r')dr'$ we obtain:

$$R = \int_{-\infty}^{x} f(x')dx' = F(x)$$

with $f(x) = \frac{dF(x)}{dx}$ being the probability density function *p.d.f* (as defined before) with:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$
$$f(\infty) = f(-\infty) = 0$$

Examples (assuming we have random numbers R_j uniformly distributed in $[0, 1]$):

- **linear** *p.d.f*: $f(x) = 2x$.
  The primitive function $F(x)$ is:

$$\begin{aligned} F(x) &= \int_0^x f(t)dt = \int_0^x 2tdt = x^2 \\ R &= F(x) = x^2 \\ x_j &= \sqrt{R_j} \end{aligned}$$

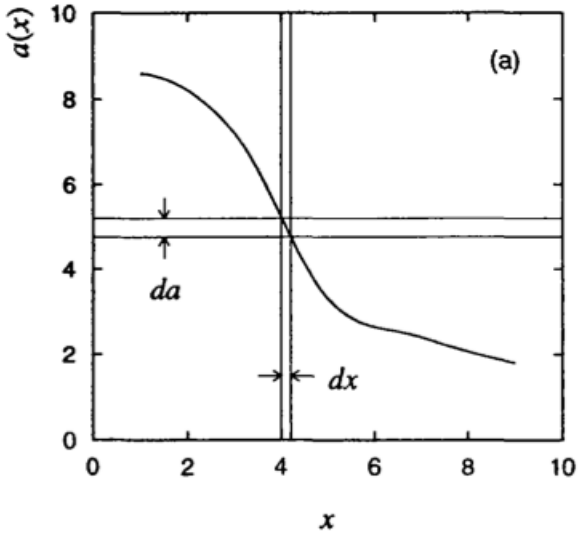

Figure 2.4: Illustration of $a(r)dr = f(x)dx$. Picture from [4] [p14]

For any uniformly distributed random numbers R_j, the x_j values are distributed according to the function $f(x) = 2x$, when calculated as $x_j = \sqrt{R_j}$

- **exponential** *p.d.f*: $f(x, \lambda) = \lambda \exp(-\lambda x)$.
  The primitive function $F(x)$ is:

$$\begin{aligned} F(x) &= \int_0^x f(t)dt = \int_0^x \lambda e^{(-\lambda t)}dt = \lambda \left. \frac{-1}{\lambda} e^{(-\lambda t)} \right|_0^x \\ &= 1 - e^{-\lambda x} \\ -R + 1 &= e^{-\lambda x} \\ \log(1 - R) &= -\lambda x \\ x_j &= \frac{-1}{\lambda} \log(R_j) \end{aligned}$$

  The values x_j can be generated from a uniform distribution of random numbers R_j with $x_j = \frac{-1}{\lambda} \log(1 - R_j) = \frac{-1}{\lambda} \log(R_j)$ since for a uniform distribution the probability of occurrence of $1 - R_j$ is the same as for R_j

- *p.d.f:* $f'(x) = 1/x$ in the range $[x_{min}, x_{max}]$
  The normalization integral is:

$$\int_{x_{min}}^{x_{max}} \frac{1}{t} dt = \log \frac{x_{max}}{x_{min}} \tag{2.22}$$

Since this function $f'(x)$ is not normalized to unity, the normalization factor has to be included:

$$f(x) = \frac{f'(x)}{\log \frac{x_{max}}{x_{min}}} = \frac{1}{x} \frac{1}{\log \frac{x_{max}}{x_{min}}} \tag{2.23}$$

The primitive function $F(x)$ is then:

$$\begin{aligned} F(x) &= \int_{x_{min}}^{x} f(t)dt \\ &= \frac{1}{\log \frac{x_{max}}{x_{min}}} \int_{x_{min}}^{x} \frac{1}{t} dt = \frac{1}{\log \frac{x_{max}}{x_{min}}} \log \frac{x}{x_{min}} \\ R &= \frac{\log \frac{x}{x_{min}}}{\log \frac{x_{max}}{x_{min}}} \\ \log \left(\frac{x_{max}}{x_{min}} \right)^{R} &= \log \left(\frac{x}{x_{min}} \right) \end{aligned} \tag{2.24}$$

The values x_j can be generated from a uniform distribution of random numbers R_j with $x_j = x_{min} \left(\frac{x_{max}}{x_{min}} \right)^{R_j}$.

- **brute force** or **hit and miss method**:
  If there is no easy way to find an analytically integrable function, which can be inverted one can use the *hit-and-miss* method. Assume we want to generate random numbers according to a function $f(x)$ in the interval $[a, b]$. The procedure is then the following: determine the maximum value, the function $f(x)$ can reach in $[a, b]$, which is f_{max}. Then select x_i uniformly in the range $[a, b]$ with $x_i = a + (b - a)R_i$ with R_i in $(0, 1)$. Use another random variable R_j also in $(0, 1)$. Decide according to the following, if the pair R_i, R_j of random numbers is accepted or rejected.

  $$\begin{aligned} \text{if } f(x_i) < R_j \cdot f_{max} &\quad \rightsquigarrow \quad \text{reject} \\ \text{if } f(x_i) > R_j \cdot f_{max} &\quad \rightsquigarrow \quad \text{accept} \end{aligned}$$

  The accepted random numbers x_i follow then exactly the distribution of the function $f(x)$. The only disadvantage of this method is, that depending on the function $f(x)$, it can be rather inefficient.

- improvements of the *hit and miss method*.
  Find a function $g(x)$ which is similar to $f(x)$ but which is integrable and invertible, i.e. $G(x) = \int g(x)dx$ and $G^{-1}(x)$ must exist. Then choose a constant such that always $c \cdot g(x) > f(x)$ for all x. Generate x according to the function $g(x)$ with the methods described above. Generate another random variable R_j and apply the *hit and miss method* as above:

  $$\begin{aligned} \text{if } f(x_i) < R_j \cdot c \cdot g(x) &\quad \rightsquigarrow \quad \text{reject} \\ \text{if } f(x_i) > R_j \cdot c \cdot g(x) &\quad \rightsquigarrow \quad \text{accept} \end{aligned}$$

The accepted distribution of variables x_i will follow the original function $f(x)$.

2.4 Law of Large Numbers and Central Limit Theorem

The law of large numbers is fundamental for all the considerations above [4, 10, 11]. The law says, that for uniformly distributed random values u_i in the interval $[a, b]$ the sum of the probability density functions converges to the true estimate of the mean of the function $f(x)$:

$$\frac{1}{N}\sum_{i=1}^{N} f(u_i) \rightsquigarrow \frac{1}{b-a}\int_a^b f(u)du \tag{2.25}$$

The law of large numbers has been applied in the sections before implicitly. The function $f(x)$ must satisfy certain conditions: it must be integrable, and it must be finite in the whole range of $[a, b]$. The left hand side of eq.(2.25) is just a Monte Carlo estimate of the integral on the right hand side and the law of large numbers says that the Monte Carlo estimate of the integral is a consistent estimate of the true integral as the size of the random sample becomes large. At this stage, nothing is said, how large "large" has to be.
The law of large numbers tells that for infinitely large numbers the Monte Carlo estimate of the integral converges to the true estimate of the integral. The Central Limit Theorem tells us how the convergence goes for finite number of N. The Central Limit Theorem says that the sum of a large number of random variables follows a normal distribution (that is the sum of random variables is Gauss distributed) no matter according to which *p.d.f* the individual random variables were generated, only the number N must be large enough and the random variables must have finite expectation values and variances. An example of the application of the Central Limit Theorem is the construction of a Random Number generator for Gaussian distributed random numbers, which will be done in the exercises:

- take a sum of uniformly distributed random numbers R_i:

$$R_n = \sum_{i=1}^{n} R_i$$

The expectation value and the variance are calculated according to the rules in eq.(2.8,2.11):

$$\begin{aligned} E[R_1] &= \int u du = \frac{1}{2} \\ V[R_1] &= \int \left(u - \frac{1}{2}\right)^2 du = \frac{1}{12} \\ E[R_n] &= \frac{n}{2} \\ V[R_n] &= \frac{n}{12} \end{aligned}$$

According to the Central Limit Theorem the sum of random values is Gauss distributed. To obtain a distribution centered around 0 with $\sigma = 1$ we take:

$$\frac{\sum_i x_i - \sum_i \mu_i}{\sqrt{\sum_i \sigma_i^2}} \to \mathcal{N}(0,1)$$

For example we sum $n = 12$ random numbers (many times $N \to \infty$) and we obtain a "normal" (Gauss) distribution $\mathcal{N}$ [11]:

$$\mathcal{N}(0,1) \to \frac{R_n - n/2}{\sqrt{n/12}} = R_{12} - 6$$

2.5 Monte Carlo Integration

Already in the previous sections we had to deal with the problem to obtain a reliable estimate of the true value of an integral [9]:

$$I = \int_a^b f(x)dx$$

The integral I is directly connected to the expectation value of the function $f(x)$ with the x values distributed according to a probability density function $g(x)$.

$$E[f] = \int_{-\infty}^{\infty} f(x)g(x)dx$$

where the *p.d.f.* $g(x)$ must be defined such, that it vanishes outside the range of (a, b). In the case of uniformly distributed x this reduces to $g(x) = 1/(b-a)$ for $a < x < b$ (and $g(x) = 0$ otherwise) which gives:

$$E[f] = \int_{-\infty}^{\infty} f(x)g(x)dx = \frac{1}{b-a}\int_a^b f(x)dx$$

The Monte Carlo estimate of the integral is then:

$$I \approx I_{MC} = (b-a)\frac{1}{N}\sum_{i=1}^{N} f(x_i) \tag{2.26}$$

and the variance is:

$$V[I_{MC}] = \sigma_I^2 = V\left[(b-a)\frac{1}{N}\sum_{i=1}^{N} f(x_i)\right] \tag{2.27}$$

$$= \frac{(b-a)^2}{N^2} V\left[\sum_{i=1}^{N} f(x_i)\right] \tag{2.28}$$

$$= \frac{(b-a)^2}{N} V[f] \tag{2.29}$$

The variance depends on the number of times the integrand is evaluated, but also on the variance of f: $V[f]$.
Applying the definition of the variance eq.(2.11), the variance $V[f]$ becomes (with $\bar{f} = \int f dx = 1/N \sum f_i$ and assuming $g(x)$ being uniform):

$$\begin{aligned} V[f] &= \int (f - \bar{f})^2 g dx = \int (f^2 - 2f\bar{f} + \bar{f}^2) g dx && (2.30) \\ &= \int f^2 g dx - \bar{f}^2 && (2.31) \\ &= \sum \frac{f_i^2}{N} - \left(\frac{\sum f_i}{N} \right)^2 && (2.32) \\ & && (2.33) \end{aligned}$$

Then the $V[I]$ becomes:

$$V[I] = \frac{1}{N}(b-a)^2 \left(\frac{1}{N} \sum f_i^2 - \left(\frac{\sum f_i}{N} \right)^2 \right)$$

With this we can estimate the uncertainty of a Monte Carlo integration (use this in the exercises).
The Monte Carlo integration gives a probabilistic uncertainty band: we can only give a probability that the MC estimate lies within a certain range of the true values [3].
To further improve the accuracy of the Monte Carlo integration, several approaches exist:

- **importance sampling**
  If an approximate function $g(x)$ exists then the integral I can be estimated to:

$$\begin{aligned} I = \int_a^b f(x) dx &= \int_a^b \frac{f(x)}{g(x)} g(x) dx \\ &= \int h(x) g(x) dx \\ &= E\left[\frac{f(x)}{g(x)} \right] \end{aligned}$$

  provided $g(x)$ is normalized and integrable in $[a, b]$. Thus the integration reduces to calculating the expectation value of $E[f/g]$, if the values of x are distributed according the *p.d.f* $g(x)$. The values of x can be generated according to the methods discussed in the previous sections and we obtain:

$$I = \frac{1}{N} \sum \frac{f(x_i)}{g(x_i)} \quad (2.34)$$

We assume that $g(x)$ is a *p.d.f* normalized to 1 in the integration range. For example using $g(x) = (1/x)1/\log\left(\frac{x_{max}}{x_{min}}\right)$ (see eq.(2.23)) gives then:

$$I = \frac{\log\left(\frac{x_{max}}{x_{min}}\right)}{N} \sum \frac{f(x_i)}{\frac{1}{x_i}}. \tag{2.35}$$

The variance is then given by:

$$V\left[\frac{f(x)}{g(x)}\right] = E\left[\left(\frac{f(x)}{g(x)} - E\left[\frac{f(x)}{g(x)}\right]\right)^2\right] \tag{2.36}$$

A danger in this method is when $g(x)$ becomes zero or approaches zero quickly [3].

- **subtraction method (control variates)** [3]
  Find a function $g(x)$ which is close to the true function $f(x)$:

$$\int_a^b f(x)dx = \int_a^b g(x)dx + \int_a^b (f(x) - g(x))\, dx$$

  This method also reduces the variances and is especially successful if the function $f(x)$ has a divergent part. This method is often used in NLO QCD calculations.

- **stratified sampling**
  divide the integration region into subintervals:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{2.37}$$

  Then the integral is:

$$I = \frac{c-a}{n/2} \sum_1 f_i + \frac{b-c}{n/2} \sum_2 f_i \tag{2.38}$$

  with the variance (if we take $c - a = b - c = (a - b)/2$):

$$\begin{aligned} V[I] &= V[I_1] + V[I_2] \\ &= \frac{(b-a)^2}{N}\left(\frac{\sum_1 f_i^2}{N} + \frac{\sum_2 f_i}{N} - 2\left[\left(\frac{\sum_1 f_i}{N}\right)^2 + \left(\frac{\sum_2 f_i}{N}\right)^2\right]\right) \end{aligned}$$

  We obtain a smaller variance, since the fluctuations in each interval are smaller.

- **brute force method**
  The accept-reject method also works for MC integration. Defining I_0 as the area in $[a, b]$ and f_{max} as the maximum of the function $f(x)$ in this range. With a random

number R_i we generate x_i and another random number R_j is used to accept or reject the pair of random numbers i, j according to:

$$\begin{aligned} \text{if } f(x_i) < R_j \cdot f_{max} &\quad \rightsquigarrow \quad \text{reject} \\ \text{if } f(x_i) > R_j \cdot f_{max} &\quad \rightsquigarrow \quad \text{accept} \end{aligned}$$

We count the number of trails with N_0 and the number of accepted events with N. Then we obtain:

$$\begin{aligned} I &= \int_a^b f(x)dx \\ &= I_0 \frac{N}{N_0} \end{aligned}$$

The variance $V[r] = (\delta(N))^2 = \sigma^2$ is (using binomial statistics with $E[r] = N_0 P$ and $V[r] = N_0 P(1-P)$ with $P = N/N_0$):

$$V[r] = N(1-P)$$

With this we can calculate the uncertainty of the integral estimate $\delta(I)$ as:

$$\frac{\delta I}{I} = \frac{I_0 \sigma / N_0}{I_0 N / N_0} = \sqrt{\frac{N(1-P)}{N^2}}$$

Chapter 3

Parton evolution equation

Here we will derive the evolution equation for the parton densities in the collinear (small t) limit, the so called DGLAP evolution equations (named after the authors Dokshitzer, Gribov, Lipatov, Altarelli, Parisi [12–15]). The expression for the deep inelastic scattering cross section (or the structure function F_2) including $\mathcal{O}(\alpha_s)$ corrections is given by:

$$\begin{aligned}\frac{\sigma^{\gamma^* p}}{\sigma_0} = \frac{F_2}{x} &= \sum e_q^2 \int \frac{d\xi}{\xi}\left(q(\xi,\mu^2)\left[\delta\left(1-\frac{x}{\xi}\right)+\frac{\alpha_s}{2\pi}P_{qq}\left(\frac{x}{\xi}\right)\log\left(\frac{Q^2}{\mu^2}\right)\right]\right. \\ &\left. +g(\xi,\mu^2)\left[\frac{\alpha_s}{2\pi}P_{qg}\left(\frac{x}{\xi}\right)\log\left(\frac{Q^2}{\mu^2}\right)\right]\right) \qquad (3.1)\end{aligned}$$

The cross section for small transverse momenta (or at small t) is divergent, and therefore gives a dominant contribution to the total cross section. For the price of a scale dependent parton density we have moved the divergent behavior into the bare (and not observable) parton densities, with the result that then the expression were finite (a procedure called renormalization). Since the γp cross section $\sigma^{\gamma^* p}$ (or equivalently the structure function F_2) as an observable cannot depend on the arbitrary scale μ^2, we tmust require, that it is μ^2-scale independent. This is satisfied by the requirement

$$\frac{\partial F_2}{\partial \mu^2} = 0$$

Using eq.(3.1) (for simplicity we treat here only the quark part, the gluon part is treated similarly) we obtain:

$$\begin{aligned}\frac{\delta F_2}{\delta \mu^2} &= \int \frac{d\xi}{\xi}\left(\frac{\partial q(\xi,\mu^2)}{\partial \mu^2}\left[\delta\left(1-\frac{x}{\xi}\right)+\frac{\alpha_s}{2\pi}P_{qq}\left(\frac{x}{\xi}\right)\log\left(\frac{Q^2}{\mu^2}\right)\right]\right. \qquad (3.2)\\ &\left. +q(\xi,\mu^2)\frac{\alpha_s}{2\pi}P_{qq}\left(\frac{x}{\xi}\right)\frac{\partial}{\partial\mu^2}\left[\log Q^2-\log\mu^2\right]\right) \\ &= \frac{\partial q(x,\mu^2)}{\partial \mu^2}+\int\frac{d\xi}{\xi}\frac{\alpha_s}{2\pi}P_{qq}\left(\frac{x}{\xi}\right)\log\frac{Q^2}{\mu^2}\frac{\partial q(\xi,\mu^2)}{\partial\mu^2} \qquad (3.3)\end{aligned}$$

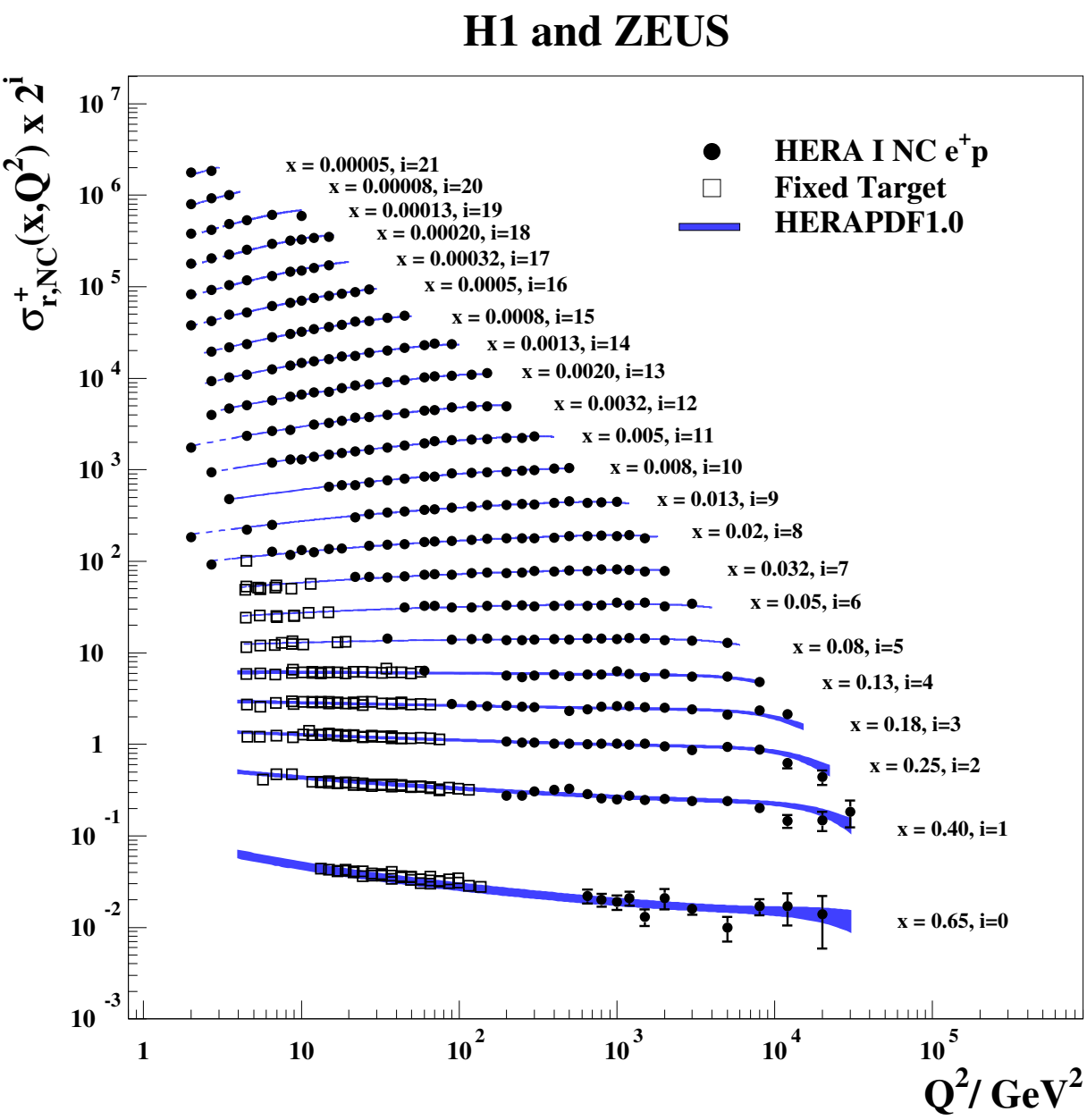

Figure 3.1: The structure function $F_2(x, Q^2)$ as a function of Q^2 as measured in DIS scattering $e + p \to e' + X$ at HERA [16].

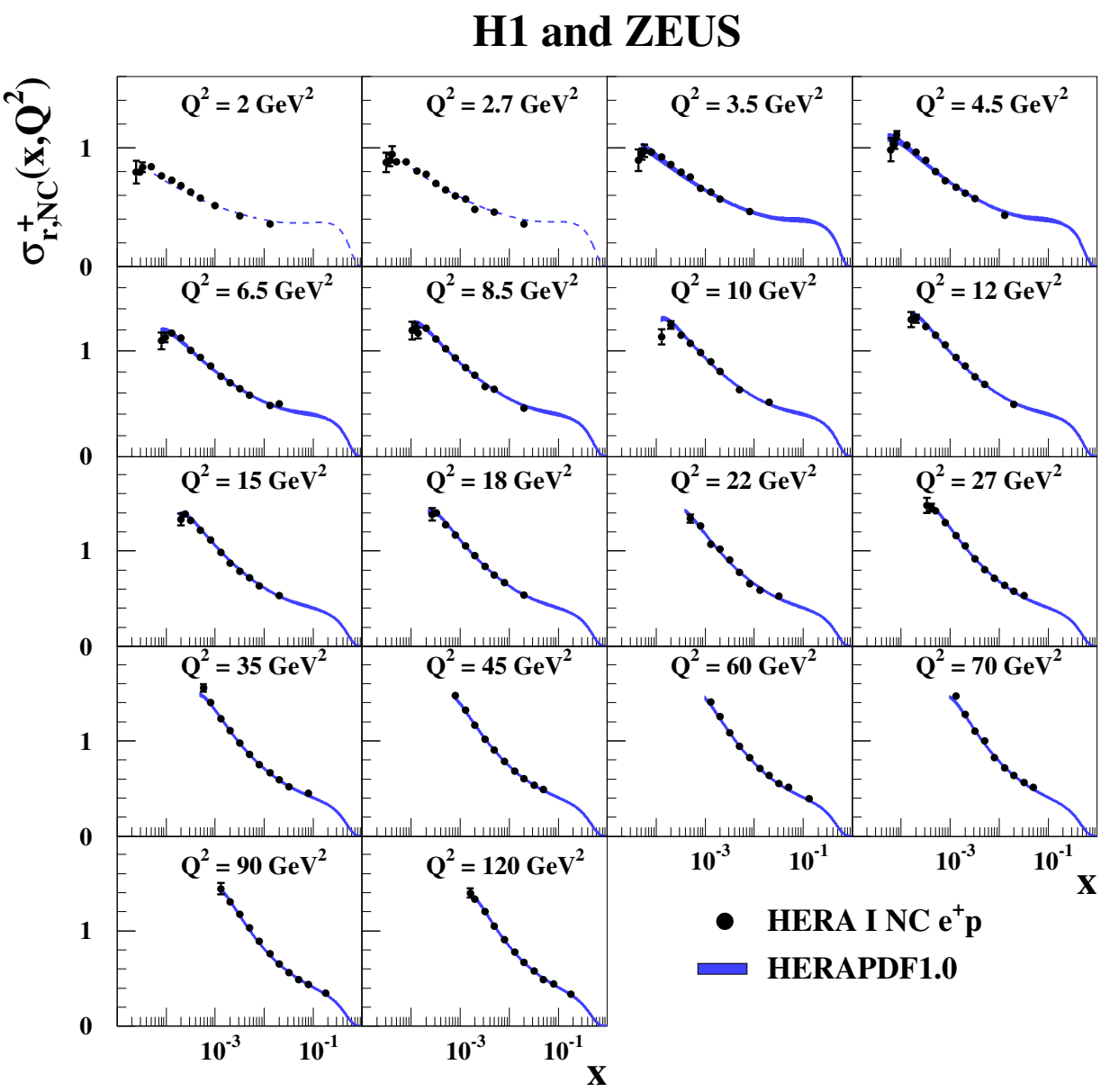

Figure 3.2: The structure function $F_2(x, Q^2)$ as a function of x as measured in DIS scattering $e + p \to e' + X$ at HERA [16].

$$+\int \frac{d\xi}{\xi} q(\xi,\mu^2)\frac{\alpha_s}{2\pi}P_{qq}\left(\frac{x}{\xi}\right)\left(-\frac{1}{\mu^2}\right)$$

Now we collect all terms of $\mathcal{O}(\alpha_s)$ (note the second term in eq.(3.3) is of $\mathcal{O}(\alpha_s^2)$ and therefore does not contribute) and obtain:

$$\frac{dq_i(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[q_i(\xi,\mu^2)P_{qq}\left(\frac{x}{\xi}\right)\right] \tag{3.4}$$

Including also the gluon part we obtain:

$$\frac{dq_i(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[q_i(\xi,\mu^2)P_{qq}\left(\frac{x}{\xi}\right)+g(\xi,\mu^2)P_{qg}\left(\frac{x}{\xi}\right)\right] \tag{3.5}$$

and similarly for the gluons

$$\frac{dg(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[\sum_i q_i(\xi,\mu^2)P_{gq}\left(\frac{x}{\xi}\right)+g(\xi,\mu^2)P_{gg}\left(\frac{x}{\xi}\right)\right] \tag{3.6}$$

The splitting functions are given by:

$$P_{qq} = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right) \tag{3.7}$$

$$P_{gq} = \frac{4}{3}\left(\frac{1+(1-z)^2}{z}\right) \tag{3.8}$$

$$P_{qg} = \frac{1}{2}\left(z^2+(1-z)^2\right) \tag{3.9}$$

$$P_{gg} = 6\left(\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right) \tag{3.10}$$

Eq.3.5 and 3.6 are the DGLAP evolution equations in leading order of $\alpha_{\rm s}$. They describe the evolution of the parton density with the scale μ^2. By knowing the parton density at any scale μ^2, these equations predict the parton density at any other scale. Although we cannot calculate the parton densities from first principles, these equations allow us to predict the parton densities at any scale, once they are determined at another scale. In Fig. 3.1-3.2 is shown the comparison of the measurement of the structure function $F_2(x,Q^2)$ with the prediction from a DGLAP evolution. The prediction agrees with the measurement remarkably well over several orders of magnitude in x and Q^2. This is a real triumph of the theory.

3.1 Conservation and Sum Rules

In the following we investigate further the evolution equations.

3.1.1 Flavor Conservation

The scale dependent quark density as a function of the bare parton density q_0 and the scale dependent divergent part ($\kappa \to 0$) can be written as:

$$\begin{aligned} q(x,\mu^2) &= \int_x^1 \frac{d\xi}{\xi} q_0(\xi) \left[\delta(1-\frac{x}{\xi}) + \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \log\frac{\mu^2}{\kappa^2} + \dots \right] && (3.11) \\ &= \int_x^1 \frac{d\xi}{\xi} q_0(\xi) \hat{q}(z,\mu^2) + \dots && (3.12) \\ &= \int_x^1 d\xi \int_0^1 dz \delta(x - z\xi) q_0(\xi) \hat{q}(z,\mu^2) + \dots && (3.13) \end{aligned}$$

with

$$\hat{q}(z,\mu^2) = \delta(1-z) + \frac{\alpha_s}{2\pi} P_{qq}(z) \log\frac{\mu^2}{\kappa^2} \tag{3.14}$$

where we have used $z = \frac{x}{\xi}$ and $\delta(1-z)dz = \delta(1-\frac{x}{\xi})dz = \xi\delta(\xi - x)dz$.
However, this is not the full expression in $\mathcal{O}(\alpha_s)$, since we have not yet included virtual gluon radiation, self-energy insertions on the quark leg and vertex corrections. One can calculate the virtual corrections explicitly, but here we use the argument of conservation of quark (and baryon) number: the integral over z of the quark distribution cannot vary with μ^2:

$$\int_0^1 dz\, \hat{q}(z,\mu^2) = 1 \tag{3.15}$$

For this we redefine the splitting function as:

$$P_{qq}(z) = \hat{P}_{qq}(z) + k \cdot \delta(1-z) \tag{3.16}$$

With this we get:

$$\int dz \left[\delta(1-z) + \frac{\alpha_s}{2\pi} \left(\hat{P}_{qq}(z) + k \cdot \delta(1-z) \right) \log\frac{\mu^2}{\kappa^2} \right] = 1$$

With $\log\frac{\mu^2}{\kappa^2} \neq 0$ we obtain

$$\int_0^1 dz \frac{\alpha_s}{2\pi} \left(\hat{P}(z) + k \cdot \delta(1-z) \right) = 0$$

Some of the splitting functions are divergent for $z \to 1$ and we cannot perform the integral easily. However we note, that $z \to 1$ reduces to no-emission and this has a final state similar to a virtual contribution to the no-emission diagram. To treat this singularity formally we introduce a " $+$ " distribution (similar to the δ function which is only defined inside an integral):

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} = \int_0^1 dx \frac{f(x) - f(1)}{(1-x)} \tag{3.17}$$

or in general [17]:

$$\begin{aligned}\int_0^1 dx f(x)\,[F(x)]_+ &= \int_0^1 dx\,(f(x)-f(1))\,F(x)\\ \text{with}\int_0^1 dx\,[F(x)]_+ &= 0\end{aligned}$$

We now use the expression for the quark splitting $\hat{P}_{qq}(z)=\frac{1+z^2}{(1-z)_+}$:

$$\begin{aligned}\int_0^1 dz P_{qq}(z) &= \int_0^1 dz\,\left[\frac{1+z^2}{(1-z)_+}+k\cdot\delta(1-z)\right] &(3.18)\\ &= \int_0^1 dz\,\frac{1+z^2-2}{1-z}+k &(3.19)\\ &= k+\int_0^1 dz\frac{-(1-z^2)}{1-z} &(3.20)\\ &= k-\int_0^1 dz\frac{(1+z)(1-z)}{1-z} &(3.21)\\ &= k-\int_0^1 dz(1+z)=k-\frac{3}{2} &(3.22)\end{aligned}$$

where in eq.(3.19) the expression $f(z)-f(1)=(1-z)^2-2$ has been used. Thus we obtain:

$$P_{qq}(z)=\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z) \tag{3.23}$$

With this expression for P_{qq} we ensure that soft singularities are properly cancelled. This expression is essential to ensure that the sum rules are fulfilled (here for the proton case) independent of μ^2:

$$\begin{aligned}\int_0^1 dx\,u_v(x,\mu^2) &= 2\\ \int_0^1 dx\,d_v(x,\mu^2) &= 1\end{aligned}$$

In Fig. 3.3 the different diagrams which contribute to $F_2(x,Q^2)$ at $\mathcal{O}(\alpha_s)$ are shown.

3.2 Solution of DGLAP equations

Several methods exist to solve the DGLAP equations, here we only consider a numerical solution of the integro-differential equations. We first consider a solution of the evolution equation at small x and then discuss the more general case.

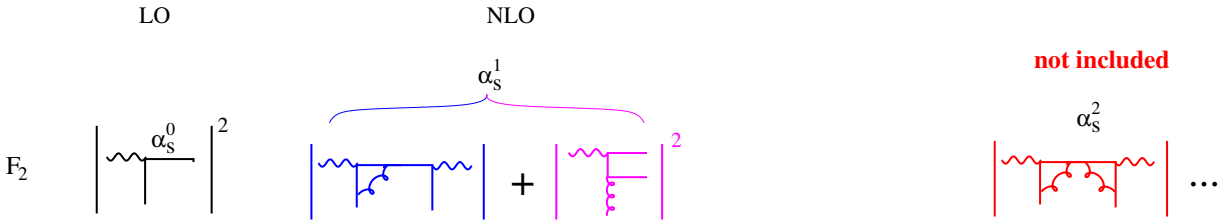

Figure 3.3: The different diagrams which contribute to $F_2(x, Q^2)$ at $\mathcal{O}(\alpha_s)$. Note that at $\mathcal{O}(\alpha_s)$ only the interference diagram of $\mathcal{O}(\alpha_s^0)$ and the virtual contribution together with the real $\mathcal{O}(\alpha_s)$ diagram contribute, while the virtual diagram squared would give $\mathcal{O}(\alpha_s^2)$.

3.2.1 Double Leading Log approximation for small x

In this section we consider only the limit of small x. In this limit, only the gluon density contributes with the splitting function $P_{gg}(x) \to 6/x$. All other contributions are small and can be neglected. With this the evolution equation eq.(3.6) becomes:

$$\frac{dg(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[g(\xi,\mu^2)P_{gg}\left(\frac{x}{\xi}\right)\right] \tag{3.24}$$

This equation can be integrated to give:

$$xg(x,\mu^2) = xg(x,\mu_0^2) + \frac{\alpha_s}{2\pi}\int_{\mu_0^2}^{\mu^2}\frac{d\mu'^2}{\mu'^2}\int_x^1\frac{d\xi}{\xi}xg(\xi,\mu^{2'})P(\frac{x}{\xi}) \tag{3.25}$$

$$= xg(x,\mu_0^2) + \frac{3\alpha_s}{\pi}\int_{\mu_0^2}^{\mu^2}\frac{d\mu'^2}{\mu'^2}\int_x^1\frac{d\xi}{\xi}\xi g(\xi,\mu^{2'}) \tag{3.26}$$

This equation is an integral equation of Fredholm type

$$\phi(x) = f(x) + \lambda\int_a^b K(x,y)\phi(y)dy$$

and can be solved by iteration (Neumann series):

$$\begin{aligned}
\phi_0(x) &= f(x)\\
\phi_1(x) &= f(x) + \lambda\int_a^b K(x,y)f(y)dy\\
\phi_2(x) &= f(x) + \lambda\int_a^b K(x,y_1)f(y_1)dy_1 + \lambda^2\int_a^b\int_a^b K(x,y_1)K(y_1,y_2)f(y_2)dy_2dy_1
\end{aligned}$$

This can be written in a compact form:

$$\phi_n(x) = \sum_{i=0}^{n}\lambda^i u_i(x) \tag{3.27}$$

with

$$\begin{aligned} u_0(x) &= f(x) \\ u_1(x) &= \int_a^b K(x,y)f(y)dy \\ u_n(x) &= \int_a^b \cdots \int_a^b K(x,y_1)K(y_1,y_2)\cdots K(y_{n-1},y_n)f(y_n)dy_1\cdots dy_n \end{aligned}$$

with the solution:

$$\phi(x) = \lim_{n\to\infty} q_n(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \lambda^i u_i(x) \tag{3.28}$$

Applying this method to solve the evolution equation for the gluon density at small x eq.(3.26) with $xg(x,\mu_0^2) = xg_0(x) = C$, we obtain:

$$xg_1(x,t) = \frac{3\alpha_s}{\pi} C \int_{t_0}^{t} d\log t' \int_x^1 d\log\xi = \frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}C \tag{3.29}$$

$$xg_2(x,t) = \frac{1}{2}\frac{1}{2}\left(\frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}\right)^2 C \tag{3.30}$$

$$\vdots$$

$$xg_n(x,t) = \frac{1}{n!}\frac{1}{n!}\left(\frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}\right)^n C \tag{3.31}$$

$$xg(x,t) = \lim_{n\to\infty}\sum_n \frac{1}{n!}\frac{1}{n!}\left(\frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}\right)^n C \tag{3.32}$$

Using the modified Bessel function:

$$I_0(z) = \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{(k!)^2} \sim e^z \tag{3.33}$$

We identify

$$z = 2\sqrt{\frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}}$$

to obtain:

$$xg(x,t) \sim C\exp\left(2\sqrt{\frac{3\alpha_s}{\pi}\log\frac{t}{t_0}\log\frac{1}{x}}\right) \tag{3.34}$$

This result has been obtained by taking the limit of double leading logarithms:

- small x limit in the splitting function which leads to $\log 1/x$
- small t limit to obtain evolution equation, which leads to $\log t$.

The DLL solution of the evolution equations results in a rapid rise of the gluon density at small x, however only so-called contributions from strongly ordered (decreasing) values of x and strongly ordered (increasing) values of t are considered.

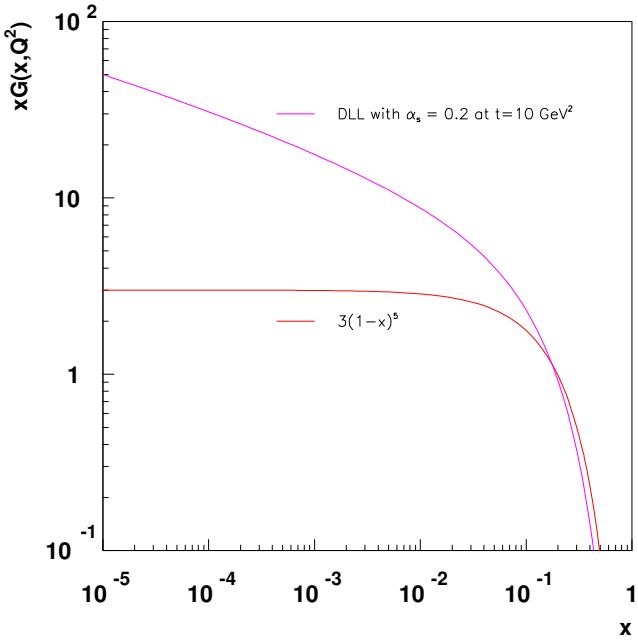

Figure 3.4: The gluon density from $xG(x) = 3(1-x)^5$ and the DLL result with $\alpha_s = 0.2$ and $t = 10\,\mathrm{GeV}^2$.

3.2.2 From evolution equation to parton branching

In the previous section we have seen how to solve the evolution equation iteratively. By performing the small x limit, we avoided the difficulties with the soft divergencies at large x; we did not need to use the plus-prescription of the splitting function. Here we now discuss a different way to treat the soft limit. The divergency of a soft real emission is cancelled by virtual contributions, that is, we can define a "resolvable" branching, which is a splitting of one into two partons, where at least in principle we can resolve the splitting. The "non-resolvable" branching consist of a contribution without branching and the virtual contributions. A detailed discussion of the parton evolution can be found in [18].
We define a "Sudakov" form factor Δ_s:

$$\Delta_s(t) = \exp\left(-\int_x^{z_{max}} dz \int_{t_0}^{t} \frac{\alpha_s}{2\pi}\frac{dt'}{t'}\tilde{P}(z)\right) \tag{3.35}$$

and use the evolution equation with the "+" prescription (using $t = \mu^2$):

$$t\frac{\partial}{\partial t} f(x,t) = \int \frac{dz}{z}\,\frac{\alpha_s}{2\pi} P_+(z)\, f\left(\frac{x}{z}, t\right)$$

Inserting the explicit expression for P_+ we obtain:

$$t\frac{\partial}{\partial t} f(x,t) = \int_0^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi} P(z)\, f\left(\frac{x}{z}, t\right) - \frac{\alpha_s}{2\pi} f(x,t)\int_0^1 dz P(z) \tag{3.36}$$

where we have used the definition in eq.(3.17):

$$\begin{aligned}\int_0^1 dz \frac{f(z)}{z} P_+(z) &= \int_0^1 dz \left(\frac{f(\frac{x}{z})}{z} - f(x)\right) P(z) \\ &= \int_0^1 dz \frac{f(\frac{x}{z})}{z} P(z) - f(x)\int_0^1 dz P(z)\end{aligned}$$

Using

$$\frac{\partial e^{-a(x)}}{\partial x} = -e^{-a(x)}\frac{\partial a(x)}{\partial x}$$

we obtain:

$$\frac{\partial \Delta_s}{\partial t} = -\Delta_s\left[\frac{1}{t}\int dz \frac{\alpha_s}{2\pi} P(z)\right] \tag{3.37}$$

$$\rightsquigarrow \quad \frac{t}{\Delta_s}\frac{\partial \Delta_s}{\partial t} = -\int dz \frac{\alpha_s}{2\pi} P(z) \tag{3.38}$$

Inserting this into eq.(3.36) we obtain:

$$t\frac{\partial}{\partial t} f(x,t) = \int \frac{dz}{z}\,\frac{\alpha_s}{2\pi} P(z)\, f\left(\frac{x}{z}, t\right) + f(x,t)\frac{t}{\Delta_s}\frac{\partial \Delta_s}{\partial t} \tag{3.39}$$

Multiplying eq.(3.39) with $1/\Delta_s$ and using $\frac{\partial}{\partial t}\frac{f}{\Delta_s} = \frac{1}{\Delta_s}\frac{\partial f}{\partial t} - \frac{f}{\Delta_s^2}\frac{\partial \Delta_s}{\partial t}$ we obtain:

$$\frac{t}{\Delta_s}\frac{\partial f(x,t)}{\partial t} - \frac{t}{\Delta_s^2} f(x,t)\frac{\partial \Delta_s}{\partial t} = \int \frac{dz}{z}\frac{1}{\Delta_s}\frac{\alpha_s}{2\pi} P(z)\, f\left(\frac{x}{z}, t\right) \tag{3.40}$$

$$t\frac{\partial}{\partial t}\frac{f(x,t)}{\Delta_s} = \int \frac{dz}{z}\frac{1}{\Delta_s}\frac{\alpha_s}{2\pi} P(z)\, f\left(\frac{x}{z}, t\right) \tag{3.41}$$

which is the DGLAP evolution equation in a form using the Sudakov form factor Δ_s as defined in eq.(3.35).
We can now integrate eq.(3.41) to obtain:

$$f(x,t) = f(x,t_0)\Delta(t) + \int \frac{dt'}{t'}\frac{\Delta(t)}{\Delta(t')}\frac{\alpha_{\rm s}(t')}{2\pi}\int \frac{dz}{z} P(z) f(\frac{x}{z}, t') \tag{3.42}$$

where we have used

$$\int_{t_0}^{t}\frac{\partial}{\partial t'}\frac{f(x,t')}{\Delta_s}dt' = \int \frac{dt'}{t'}\frac{1}{\Delta_s}\frac{\alpha_{\rm s}}{2\pi}\int \frac{dz}{z} P(z) f(\frac{x}{z}, t') \tag{3.43}$$

From eq.(3.42) we can now interpret the Sudakov form factor as being the probability for evolution without any resolvable branching from t_0 to t.
What did we gain ? We needed to treat the singularity at $z \to 1$. For this, we now introduce a upper cut-off $z_{cut} = 1 - \epsilon(\mu)$. Branchings with $z > z_{cut}$ are now classified as unresolved: they involve the emission of undetectable partons [18]. The Sudakov form factor sums virtual and real corrections to all orders; the virtual corrections affect the non-branching probability are included via unitarity: the resolvable branching probability gives via unitarity the sum of virtual and unresolvable contributions.
Eq.(3.42) can now be solved by iteration, in the same way as before. The starting function f_0 is just the first term in eq.(3.42). The first iteration f_1 involves one branching:

$$\begin{aligned} f_0(x,t) &= f(x,t_0)\Delta(t) \\ f_1(x,t) &= f(x,t_0)\Delta(t) + \frac{\alpha_{\rm s}}{2\pi}\int_{t_0}^{t}\frac{dt'}{t'}\frac{\Delta(t)}{\Delta(t')}\int_x^1 \frac{dz}{z}\tilde{P}(z) f(x/z,t_0)\Delta(t') \end{aligned} \tag{3.44}$$

The iteration is illustrated in fig.3.5 The term f_0 in eq.(3.44) is illustrated in the left part of Fig. 3.5: the evolution from t_0 to t without any resolvable branching. The term f_1 in eq.(3.44) is shown in the right part of Fig. 3.5: there is evolution from t_0 to t' without any resolvable branching, then at t' the branching happens, where the splitting is given by the splitting function $P(z)$; then the evolution continues without any resolvable branching from t' to t.
The full solution of the integral equation by iteration is then:

$$\begin{aligned} f_0(x,t) &= f(x,t_0)\Delta(t) \\ f_1(x,t) &= f(x,t_0)\Delta(t) + \frac{\alpha_{\rm s}}{2\pi}\int_{t_0}^{t}\frac{dt'}{t'}\frac{\Delta(t)}{\Delta(t')}\int \frac{dz}{z}\tilde{P}(z) f(x/z,t_0)\Delta(t') \end{aligned}$$

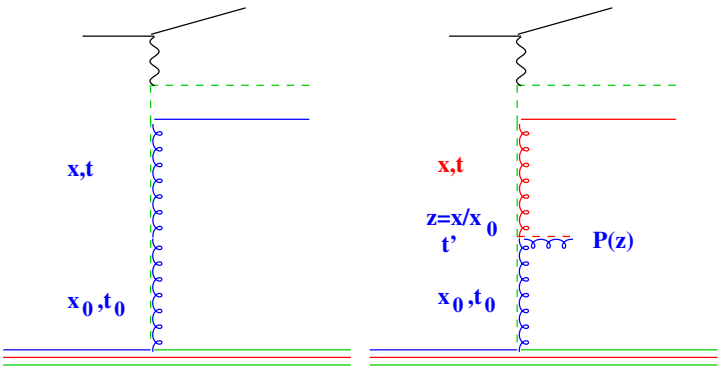

Figure 3.5: Schematic representation of the first branchings in an iterative procedure to solve the evolution equation

$$
\begin{aligned}
& = f(x,t_0)\Delta(t) + \log\frac{t}{t_0} A\otimes\Delta(t) f(x/z,t_0) \\
f_2(x,t) &= f(x,t_0)\Delta(t) + \log\frac{t}{t_0} A\otimes\Delta(t) f(x/z,t_0) + \\
& \quad \frac{1}{2}\log^2\frac{t}{t_0} A\otimes A\otimes\Delta(t) f(x/z,t_0) \\
f(x,t) &= \lim_{n\to\infty} f_n(x,t) = \lim_{n\to\infty}\sum_n \frac{1}{n!}\log^n\left(\frac{t}{t_0}\right) A^n\otimes\Delta(t) f(x/z,t_0)
\end{aligned}
\tag{3.45}
$$

where $A = \int \frac{dz}{z}\tilde{P}(z)$ is a symbolic representation of the integral over z and $\otimes$ indicates that a convolution has to be performed. The eq.(3.45) shows the solution of the DGLAP evolution equation is a resummation to all orders in $\alpha_s \log t$.[1]
The Sudakov form factor can be interpreted in terms of a probability: it is a poisson distribution with zero mean $P(0,p) = e^{-p}$. If the poisson distribution gives the probability to observe n emissions, then $P(0,p)$ gives the probability for no emission and is the so-called "non-branching probability". The one-branching probability is given in terms of Poisson statistics by: $P(1,p) = pe^{-p}$, which is exactly the first iteration of the evolution equation:

$$
\begin{aligned}
f(x,t) &= f(x,t_0)\Delta_s(t) + \int dz \int dx' \int \frac{dt'}{t'}\cdot\frac{\Delta_s(t)}{\Delta_s(t')}\frac{\alpha_s}{2\pi}\tilde{P}(z)\times \\
& \quad \times f\left(x',t_0\right)\Delta_s(t')\delta(x-zx')
\end{aligned}
\tag{3.46}
$$

where delta function has been introduced to make the different integration steps visible.

[1] It is interesting to note, that only the $1/(1-z)$ part of the splitting functions is needed in the Sudakov form factor. This simplifies the solution process.

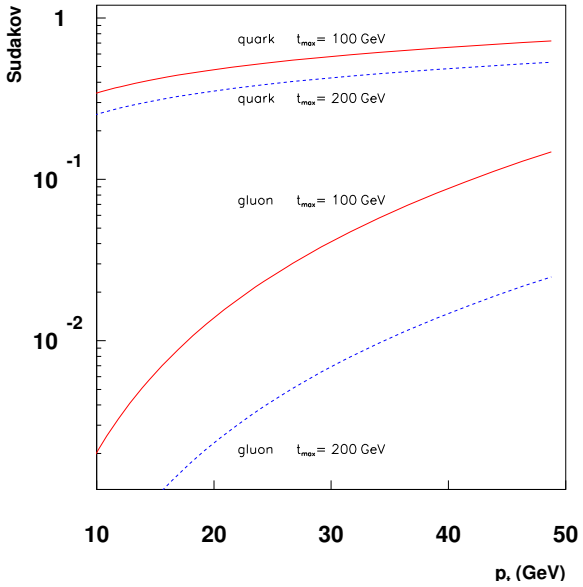

Figure 3.6: Sudakov form factor as a function of the lower scale p_t for gluon and quark splitting functions using $\alpha_s = 0.2$. The upper scale is set to $t_{max} = 100(200)$ GeV.

We have introduced a cut to avoid the divergency when $z \to 1$ via $z_{cut} = 1 - \epsilon(\mu)$, but we have not yet specified how this can be calculated. To some extend the value of z_{cut} is a matter of choice, here we give an argument based on the virtualities of the partons involved. We work in a frame, were all energies are much larger than the starting scale of the evolution Q_0. We use light-cone variables for the partons: $p^+ = 1/\sqrt{2}(E + p_z)$ and we define

$$z = \frac{p_b^+}{p_a^+}$$

being the splitting variable for a process $a \to b + c$. The light-cone vector satisfies: $p_a^2 = 2p_a^+ p_a^- - k_{ta}^2$. We work in a frame, where $k_{ta} = 0$ and $k_{tb} = -k_{tc} = k_t$. Using conservation of the "+" and "-" components of the light-cone vectors we obtain:

$$\begin{aligned} p_a^- &= p_b^- + p_c^- \\ \frac{p_a^2}{2p_a^+} &= \frac{p_b^2 + k_{tb}^2}{2p_b^+} + \frac{p_c^2 + k_{tc}^2}{2p_c^+} \\ \rightsquigarrow p_a^2 &= \frac{p_b^2 + k_t^2}{z} + \frac{p_c^2 + k_t^2}{1 - z} \end{aligned}$$

where for the last expression we have used $p_b^+ = zp_a^+$ and $p_c^+ = (1-z)p_a^+$. This equation can be rewritten to give:

$$\begin{aligned} k_t^2 &= z(1-z)p_a^2 - (1-z)p_b^2 - zp_c^2 \\ \rightsquigarrow 0 &< (1-z)Q_b^2 - zQ_c^2 \end{aligned} \tag{3.47}$$

where we have defined $Q_a^2 = -p_a^2$ and $Q_b^2 = -p_b^2$ and $Q_c^2 = p_c^2$ and $Q_c^2 > Q_0^2$, thus that parton a and b are spacelike partons while parton c is timelike. Using $(1+x)^{-m} = 1 - mx + \cdots$ we obtain from Eq.(3.47):

$$z < 1 - \frac{Q_0^2}{Q_b^2} + \cdots \tag{3.48}$$

where we have used $Q_c^2 > Q_0^2$.
In Fig. 3.6 the Sudakov form factor is shown for quark and gluon splittings for different scales t_{max} as a function of the lower scale p_t. The probability for quarks not to undergo any branching (the sudakov form factor gives the no-branching probability) is much higher than the corresponding one for gluons.

3.3 Solution of evolution equation with Monte Carlo method

AS described above, the evolution equations Eqs.(3.42) are integral equations of the Fredholm type

$$f(x) = f_0(x) + \lambda \int_a^b K(x,y) f(y) dy$$

and can be solved by iteration as a Neumann series

$$\begin{aligned} f_1(x) &= f_0(x) + \lambda \int_a^b K(x,y) f_0(y) dy \\ f_2(x) &= f_0(x) + \lambda \int_a^b K(x,y_1) f_0(y_1) dy_1 + \lambda^2 \int_a^b \int_a^b K(x,y_1) K(y_1,y_2) f_0(y_2) dy_2 dy_1 \\ &\cdots \end{aligned} \tag{3.49}$$

using the kernel $K(x,y)$, with the solution

$$f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} f_i(x). \tag{3.50}$$

In a Monte Carlo (MC) solution [19–21] we evolve from t_0 to a value t' obtained from the Sudakov factor $\Delta_s(t')$ (for a schematic visualisation of the evolution see fig. 3.7). Note that the Sudakov factor $\Delta_s(t')$ gives the probability for evolving from t_0 to t' without resolvable branching. The value t' is obtained from solving for t':

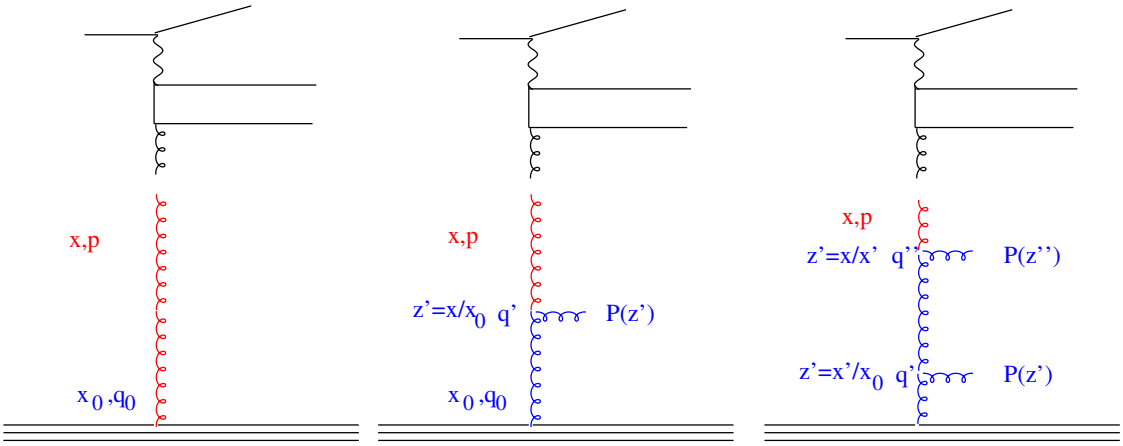

Figure 3.7: Evolution by iteration

$$R \quad = \Delta_s(t'), \tag{3.51}$$

for a random number R in $[0,1]$.
If $t' > t$ then the scale t is reached and the evolution is stopped, and we are left with just the first term without any resolvable branching. If $t' < t$ then we generate a branching at t' according to the splitting function $\tilde{P}(z')$, as described below, and continue the evolution using the Sudakov factor $\Delta_s(t'', t')$. If $t'' > t$ the evolution is stopped and we are left with just one resolvable branching at t'. If $t'' < t$ we continue the evolution as described above. This procedure is repeated until we generate t's which are larger than t. By this procedure we sum all kinematically allowed contributions in the series $\sum f_i(x,p)$ and obtain an MC estimate of the parton distribution function.
With the Sudakov factor Δ_s and using

$$\frac{\partial}{\partial t'}\Delta_s(t,t') = \frac{\partial}{\partial t'}\frac{\Delta_s(t)}{\Delta_s(t')} \quad = \quad \frac{\Delta_s(t)}{\Delta_s(t')}\left[\frac{1}{t'}\right]\int^{z_{max}} dz\tilde{P}(z),$$

we can write the first iteration of the evolution equation as

$$\begin{aligned} f_1(x,t) \quad &= \quad f_0(x,t) \\ &+ \quad \int_x^1 \frac{dz'}{z'}\int_{t_0}^t d\Delta_s(t,t')\tilde{P}(z')f_0(x/z',t')\left[\int^{z_{max}} dz\tilde{P}(z)\right]^{-1}. \end{aligned} \tag{3.52}$$

The integrals can be solved by a Monte Carlo method [11]: z is generated from

$$\int_{z_{min}}^{z} dz'\tilde{P}(z') \quad = \quad R_1\int_{z_{min}}^{z_{max}} dz'\tilde{P}(z'), \tag{3.53}$$

with R_1 being a random number in $[0,1]$, and t' is generated from

$$\begin{aligned} R_2 &= \int_{-\infty}^{x} f(x')dx' = F(x) \\ &= \int_{t'}^{t} \frac{\partial}{\partial t''}\left(\frac{\Delta_s(t)}{\Delta_s(t'')}\right) dt'' \\ &= \Delta_s(t,t') \end{aligned} \tag{3.54}$$

solving for t', using z from above and another random number R_2 in [0,1].
This completes the calculation on the first splitting. This procedure is repeated until $t' > t$ and the evolution is stopped.
With z' and t' selected according to the above the first iteration of the evolution equation yields

$$\begin{aligned} xf_1(x,t) &= xf_0(x)\Delta_s(t) \\ &+ \sum_i \tilde{P}(z_i')x_i' f_0(x_i',t_i') \left[\int^{z_{max}} dz\tilde{P}(z)\right]^{-1}, \end{aligned} \tag{3.55}$$

with $x_i' = x/z_i$.

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