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Foreword

The 24th International Colloquium on Group Theoretical Methods in Physics, also known as the "Group-24" Conference, was held from the 15th July to the 20th July 2002 in Paris, France. Although one digit short of the genuine silver jubilee, it was in many ways a remarkable and special event.

About 30 years ago a group of enthusiasts, headed by H. Bacry of Marseille and A. Janner of Nijmegen, initiated a series of annual meetings intended to provide a common forum for scientists interested in group theoretical methods, represented at that time mostly by two important communities: on the one hand, elementary particle theorists and phenomenologists, and solid state specialists, on the other, mathematicians eager to apply newly-discovered group and algebraic structures.

First four meetings took place alternatively in Marseille and Nijmegen. Soon after, the workshop acquired an international flavour, especially following the 1976 Colloquium in Montreal, Canada, and has since been held in many places around the world. It has become a bi-annual Colloquium since 1990, the year it was organized in Moscow.

This has been the first time since the foundation of this series of meetings that the Colloquium was organized once more in France. Moreover, it was held in the city of Paris, one of the cradles of group theory. Here Galois wrote his notes the night before the fatal duel; here Elie Cartan introduced the classification of semi-simple Lie algebras; one can continue *ad libitum* the list of famous mathematicians connected with this exceptional place. The very names of the streets in the Latin Quarter gave a unique flavour to our enterprise: Monge, Legendre, Lagrange, Laplace, – and many others, physicists, chemists, botanists, – the city of Paris hosted them all, and dwells on its glorious past.

Organizing a prestigious scientific meeting in a great capital is a great challenge. On the other hand, it gave the organizers a unique possibility to use its extremely rich scientific and educational infrastructure. Three Universities were associated with the organization of the Colloquium: Université Pierre et Marie Curie-Paris 6, Université Paris 7-Denis Diderot and the suburban Université de Marne-la-Vallée; they provided us with important financial support and material help. All the sessions took place in the Institutes of the Montagne Sainte Geneviève campus, just behind the Panthéon: Institut Océanographique, Institut Henri Poincaré, Institut de Chimie-Physique and Institut de Géographie. We address our special thanks to the directors of the Institut Henri Poincaré, M. Broué and A. Comtet, and to the director of Institut de Chimie-Physique Alfred Maquet, and their staff, for exceptional quality of hospitality extended to us during the Colloquium.

Important financial help has been provided by the Ministry of Research and Technology, the Ministry of Foreign Affairs and the Ministry of Education. The colloquium was also supported by French Embassies in Algeria, in Bulgaria, in Czech Republic, in Hungary, in Poland, and in Ukraine. The Wigner Medal and Hermann Weyl prize ceremonies were held at the Mairie of the 5ème arrondissement (the City Hall of the Latin Quarter in Paris) whose this end we enlarge the basic Banach-Grassmann algebra Q [13, 14, 15] to the new structure: on the classical level to the $Q_{RO} \equiv Q_R \oplus \hat{1} Q_R$ and on the quantum level $Q_{CO} \equiv Q_C \oplus \hat{\imath} Q_C$. Q_R and Q_C are real and complex Banach-Grassmann algebras [11]. The $\hat{1}$ and $\hat{\imath}$ are new elements, both of odd Grassmannian parity and moreover fulfilling relations of the form $\hat{1}^2 = 1$ and $\hat{\imath}^2 = -1$. The Q_{RO} we shall refer to as the real oddons and Q_{CO} as the complex oddons, to stress that these algebras contain the odd imaginary unit and/or the odd unit. The Q_{CO} is not a graded commutative algebra.

To define the Odd Heisenberg group [11] let us consider as an extension of the phase space $P_{(0,1)}$ by the time dimension the free $Q^{\mathbb{RO}}$-module $T_n = Q_{\mathbb{RO}}^{n|n+1}$ with the basis $\{E_i, e_i, e_0\}_{i=1}^n$, where $|e_i| = |e_0| = 0$, $|E_i| = 1$; $i = 1, 2, \ldots, n$. Let $B(\cdot\,,\cdot)$ be the odd symplectic form defined on $Q_{\mathbb{RO}}^{n,n}$ with values in $Q_{\mathbb{RO}}$.

$$v = \sum\nolimits_{i=1}^{n} p_i E_i + \sum\nolimits_{i=1}^{n} \Theta_i e_i, \quad B(v, v') = \sum\nolimits_{i=1}^{n} (p^i \Theta'_i - \Theta_i p'^i) \tag{6}$$

Now let OH_n be the set of vectors of the form

$$(v, \tau) = (p, \Theta, \tau) = (p^1, p^2, \ldots, p^n, \Theta_1, \Theta_2, \ldots, \Theta_n, \tau) \tag{7}$$

where $\tau = t \cdot \hat{1}$, $t \in Q_0^R$, $\Theta_i \in Q_1^R$, $p^i \in Q_0^{\mathbb{RO}}$. In the set OH_n we define the action in the following form

$$(v, \tau) \star (v', \tau') = (v + v', \tau + \tau' + \frac{1}{2} B(v, v')) \tag{8}$$

The$(OH_n, \star)$ is a group, we shall call it the Odd Heisenberg group. It the case of the Odd Heisenberg group we need generalization of the methods of harmonic analysis on Heisenberg group to the Grassmannian case extended by the oddons. Using odd multiplication provided in the new structure we can obtain in particular odd Grassmannian version of the Laguerre polynomials [11].

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Quantization of Phases and Moduli in terms of the Group $SO^{\uparrow}(1,2)$

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Abstract. The problem of quantizing phase and modulus associated with the polar coordinates of a 2-dimensional phase space dates back to the first years of quantum mechanics: The symplectic space $P = \{\varphi \in (-\pi, +\pi], p > 0\}$ has the global structure $S^1 \times R^+$ and cannot be quantized in the conventional naive way. The appropriate method is the group theoretical quantization which here leads to the group $SO(1,2)$ and the positive discrete series of its irreducible unitary representations. The basic classical observables $p \cos\varphi, p \sin\varphi$ and p correspond to the self-adjoint operators K_1, K_2 and K_3 of the $SO(1,2)$ Lie algebra. The approach provides appropriate quantum observables for the phase space P and a promising basis for the description of quantum optical structures

1. Introduction

The transformation

$$q(\varphi, I) = \sqrt{2I/\omega} \cos\varphi\,,\; p(\varphi, I) = -\sqrt{2\omega I} \sin\varphi\,, \tag{1}$$

is locally canonical,

$$dq \wedge dp = d\varphi \wedge dI\,, \tag{2}$$

and transforms the Hamiltonian and the Poisson bracket relation

$$H(q,p) = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2\,,\; (q,p) \in R^2\,,\; \{q,p\} = 1\,, \tag{3}$$

into

$$H = \omega I\,,\; \{q,p\} = \{\varphi, I\} = 1\,. \tag{4}$$

The phase space

$$P = \{(\varphi \in (-\pi\,\pi], I > 0)\} \tag{5}$$

of the angle and action variables φ and I has the *global* structure

$$S^1 \times R^+\,,\; R^+ = \{r \in R,\, r > 0\}\,, \tag{6}$$

which is quite different from the one we started from, namely R^2!
Dirac was the first to interpret [1] the polar decomposition of the complex amplitudes ($\bar{a}$ denotes the complex conjugate of a!)

$$a = \frac{1}{\sqrt{2}} (\sqrt{\omega} q + \frac{i}{\sqrt{\omega}} p) = I^{1/2} e^{-i\varphi}\,, \tag{7}$$

$$\bar{a} = \frac{1}{\sqrt{2}} (\sqrt{\omega} q - \frac{i}{\sqrt{\omega}} p) = I^{1/2} e^{i\varphi}\,, \tag{8}$$

$$I = \bar{a} a\,, \tag{9}$$

in terms of quantum mechanical operators $\hat{I} = N = a^+ a$ and $\hat{\varphi}$ by using a corresponding polar decomposition of the annihilation and creation operators

$$a = e^{-i\hat{\varphi}}\sqrt{N} = \sqrt{N+1}\,e^{-i\hat{\varphi}}\,,\ a^+ = \sqrt{N}e^{i\hat{\varphi}} = e^{i\hat{\varphi}}\sqrt{N+1}\,, \tag{10}$$

with the commutator

$$[\hat{\varphi}, N] = i\,. \tag{11}$$

But even before the second of Dirac's just quoted papers appeared, London had realized [2] that the commutation relation (11) cannot hold, because

$$\langle n_2|[\hat{\varphi},\hat{N}]|n_1\rangle = (n_1 - n_2)\langle n_2|\hat{\varphi}|n_1\rangle = i\,\delta_{n_2 n_1}\ ! \tag{12}$$

In a second paper [3] London suggested the use of the operators

$$E_- = aN^{-1/2}\,,\ E_+ = N^{-1/2}a^+ \tag{13}$$

instead.
That proposal, however, was not used for about 40 years, but rediscovered by Susskind and Glogower in the mid-sixties [4] and worked out in more detail by Carruthers and Nieto [5]. The subject is still controversial to-day [6]

2. Group theoretical quantization

A new promising ansatz in order to quantize the phase space (5) consistently, is group theoretical quantization, beautifully reviewed by Isham [7]. In our special context the group $SO(1,2)$ is the relevant quantizing group [8], leading to a consistent quantum theory with possible applications in quantum optics [9]. Some of its essential features are the following:
The basic classical "observables" on the phase space turn out to be the functions

$$h_1 = I\cos\varphi\,,\ h_2 = -I\sin\varphi\,,\ h_3 = I\,, \tag{14}$$

which obey the (Poisson) Lie algebra

$$\{h_3, h_1\} = -h_2\,,\ \{h_3, h_2\} = h_1\,,\ \{h_1, h_2\} = h_3\,. \tag{15}$$

This is the Lie algebra of the group $SO(1,2)$ or one of its (infinitely) many covering groups, like, e.g. $SU(1,1)$.
Quantization is implemented by employing appropriate irreducible unitary representations of the group. The 3 self-adjoint generators K_j of the corresponding 1-parameter unitary subgroups are the quantized versions of the classical observables (14):

$$K_j = \hat{h}_j\,,\ j = 1,2,3\,, \tag{16}$$

which obey the commutation relations

$$[K_3, K_1] = iK_3\,,\ \ [K_3, K_2] = iK_2\,,\ \ [K_1, K_2] = -iK_3\,. \tag{17}$$

The appropriate irreducible unitary representations are those of the positive discrete series which are characterized by the existence of a (ground) state $|k,0\rangle$ for which

$$K_-|k,0\rangle = 0\,,\ K_- = K_1 - iK_2\,. \tag{18}$$

The generator K_3 of the compact subgroup has the spectrum

$$\mathrm{spec}(K_3) = \{k+n,\, k>0, n = 0,1,\ldots\} \tag{19}$$

The positive parameter k, which characterizes an irreducible unitary representation, can take the values $k = 1,2,\ldots$ for the group $SO(1,2)$ and the values $k = 1/2, 1, 3/2, 2, \ldots$ for $SU(1,1)$. Taking the expectation values of the operators K_1 and K_2 with respect to the different coherent states associated with $SO(1,2)$ all of which are characterized by a complex number shows that these operators indeed "measure" the cos and the sin of the phase of those complex numbers.

3. Harmonic oscillator

One may illustrate many features of the generalized cos- and sin-operators K_1 and K_2 by means of the harmonic oscillator. A reason is that one can construct a realization of the $SU(1,1)$ Lie algebra generators from the oscillator annihilation and creation operators a and a^+:

$$K_- = \sqrt{N+2k}\,a\,,\ K_+ = (K_-)^+\,,\ K_3 = a^+a + k = N + k\,, \tag{20}$$

More interesting is the inverse:
Given the self-adjoint operators $K_\pm, K_3$ of a representation characterized by k, then one can define annihilation and creation operators - and therefore operators $\hat{q}$ and $\hat{p}$ - by solving the relations (20) for a and a^+!
Another point of interest is the following: For $k = 1/2$ we can identify the Hamiltonian H of the harmonic oscillator with the operator ωK_3. But for $k = 1/2$ the group $SU(1,1)$ has the following explicit irreducible unitary representation

$$(f_2, f_1) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \bar{f}_2(\varphi) f_1(\varphi)\,, \tag{21}$$

$$|k = 1/2, n\rangle = e^{in\varphi}\,, n = 0,1,\ldots\,, \tag{22}$$

$$K_3 = \frac{1}{i}\,\partial_\varphi + 1/2\,, \tag{23}$$

$$K_- = e^{-i\varphi}\,\frac{1}{i}\,\partial_\varphi\,, \tag{24}$$

$$K_+ = e^{i\varphi}\,(\frac{1}{i}\,\partial_\varphi + 1)\,, \tag{25}$$

$$H = \omega K_3\,. \tag{26}$$

Thus, it is possible to describe the quantum physics of the harmonic oscillator in this Hilbert space. Any element $f(\varphi)$ may be expanded in a series

$$f(\varphi) = \sum_{n=0}^{\infty} c_n e^{in\varphi}\,. \tag{27}$$

The use of this Hilbert space also allows for a critical evaluation of the usual - somewhat controversial - notion of "phase states" [5, 6]
For the coherent states

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,,\ \alpha = |\alpha|e^{i\beta}\,, \tag{28}$$

we get - using the relations (20) - the expectation values

$$\langle\alpha|K_1|\alpha\rangle = |\alpha|\cos\beta\,\langle\alpha|\sqrt{N+1}|\alpha\rangle\,, \tag{29}$$

$$\langle\alpha|K_2|\alpha\rangle = -|\alpha|\sin\beta\,\langle\alpha|\sqrt{N+1}|\alpha\rangle\,, \tag{30}$$

$$\langle\alpha|K_3|\alpha\rangle = |\alpha|^2 + 1/2\,, \tag{31}$$

where

$$\langle\alpha|\sqrt{N+1}|k,\alpha\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sqrt{n+1}\,\frac{|\alpha|^{2n}}{n!}\,. \tag{32}$$

We see here explicitly how the operators K_1 and K_2 measure the phase of the complex number α associated with the state $|\alpha\rangle$:

$$\tan\beta = -\frac{\langle K_2\rangle_\alpha}{\langle K_1\rangle_\alpha}\,. \tag{33}$$

The operators

$$K_1 = \frac{1}{2}(K_+ + K_-) = \cos\varphi\,\frac{1}{i}\partial_\varphi + \frac{1}{2}e^{i\varphi}\,, \tag{34}$$

$$K_2 = \frac{1}{2i}(K_+ - K_-) = \sin\varphi\,\frac{1}{i}\partial_\varphi + \frac{1}{2i}e^{i\varphi}\,. \tag{35}$$

have eigenvalues h_1 and h_2 and eigenfunctions

$$f_{h_1}(\varphi) = |2\cos\varphi|^{-1/2}|\tan(\varphi/2+\pi/4)|^{ih_1}e^{-i\varphi/2}\,,\ h_1 \in R\,, \tag{36}$$

$$f_{h_2}(\varphi) = |2\sin\varphi|^{-1/2}|\tan(\varphi/2)|^{ih_2}e^{-i\varphi/2}\,,\ h_2 \in R\,,\ \varphi \in (0, 2\pi)\,. \tag{37}$$

Many more details and results will be contained in a forthcoming publication [10].

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p-Mechanical Brackets and Method of Orbits

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Abstract. We use the orbit method of Kirillov to derive the p-mechanical brackets [8]. They generate the quantum (Moyal) and classic (Poisson) brackets on respective orbits corresponding to representations of the Heisenberg group. This highlights connections of p-mechanical brackets with deformation quantisation and Moyal's product. The p-brackets are invariant under automorphisms of the Heisenberg group, this leads to the symplectic invariance of quantum and classic mechanics.

AMS classification scheme numbers: 81R05, 81R15, 22E27, 22E70, 43A65

1. The Heisenberg Group and Its Representations

Let (s, x, y), where $x, y \in \mathbb{R}^n$ and $s \in \mathbb{R}$, be an element of the Heisenberg group $\mathbb{H}^n$ [3, 4]. The group law on $\mathbb{H}^n$ is given as follows:

$$(s, x, y) * (s', x', y') = (s + s' + \frac{1}{2}\omega(x, y; x', y'), x + x', y + y'), \tag{1}$$

where the non-commutativity is solely due to ω—the *symplectic form* [1, § 37] on $\mathbb{R}^{2n}$:

$$\omega(x, y; x', y') = xy' - x'y. \tag{2}$$

The Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$ is spanned by left-invariant vector fields

$$S = \frac{\partial}{\partial s}, \qquad X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial s}, \qquad Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial s} \tag{3}$$

on $\mathbb{H}^n$ with the Heisenberg *commutator relations* $[X_i, Y_j] = \delta_{i,j}S$ and all other commutators vanishing. The exponential map $\exp : \mathfrak{h}^n \to \mathbb{H}^n$ respecting the multiplication (1) and Heisenberg commutators is

$$\exp : sS + \sum_{k=1}^{n}(x_k X_k + y_k Y_k) \mapsto (s, x_1, \ldots, x_n, y_1, \ldots, y_n).$$

The adjoint representation $\mathrm{Ad} : \mathbb{H}^n \to \mathbb{H}^n$ given by $\mathrm{Ad}(g)h = g^{-1}hg$ fixes the unit $e \in \mathbb{H}^n$. The differential $\mathrm{ad} : \mathfrak{h}^n \to \mathfrak{h}^n$ of Ad at e is a linear map given by the Lie commutator: $\mathrm{ad}(A) : B \mapsto [B, A]$. The dual space $\mathfrak{h}_n^*$ to the Lie algebra $\mathfrak{h}^n$ is realised by the left invariant first order differential forms on $\mathbb{H}^n$. By the duality between $\mathfrak{h}^n$ and $\mathfrak{h}_n^*$ the map ad generates the *co-adjoint representation* [5, § 15.1] $\mathrm{ad}^* : \mathfrak{h}_n^* \to \mathfrak{h}_n^*$:

$$\mathrm{ad}^*(s, x, y) : (h, q, p) \mapsto (h, q + hy, p - hx), \quad \text{where } (s, x, y) \in \mathbb{H}^n \tag{4}$$

‡ On leave from the Odessa University.