Scale and Conformal Symmetry in Hadron Physics

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Preface

In May 1972 an informal meeting on the topic "Outlook for Broken Conformal Symmetry in Elementary Particle Physics" was organized in Frascati by the Advanced School of Physics at the National Laboratories of CNEN. The present volume contains most of the general reviews that were presented at the meeting, organized in a consistent fashion and in some parts expanded by the authors. It is hoped that the book will provide an ample and advanced account of the wide subject of scale and conformal symmetry. The broader topic, rather than strictly conformal symmetry, was chosen because of the inextricable connections between the two subjects, conformal symmetry usually being introduced as a natural extension of scale symmetry.

The interest in such symmetries arises naturally from the need for a better understanding of the crucial problem of short-distance behavior in quantum field theory. Theoretically the connection seems to be rigorously derivable, at least for certain renormalizable field theories, by study of the so-called Gell-Mann/Low limit. In its turn the requirement of conformal symmetry can act as a powerful and consistent constraint on the solutions of the Green function integral equations. Vacuum expectation values and operator product expansions are severely limited by conformal symmetry, and, interestingly, they exhibit mathematical structures reminiscent of the duality formalism. In particular, two- and three-point functions are uniquely determined, apart from the values of the coupling constants and scale dimensions.

The by-now famous SLAC experiments on inelastic electron scattering aroused wide interest in scale symmetry and its possible extensions. The notion of "exact scaling" when inserted into a conformal framework appears, however, to be quite far-reaching and demanding. The problem is that of canonical dimensions and of theories possessing additional conservation laws. Anomalies in Ward identities are independently known to appear in certain problems, raising theoretical questions of great interest.

The contributions in this volume touch upon most of the theoretical problems on which the interest of theoreticians is presently concentrated. Prospects for future work appear highly exciting; fascinating confluences of various approaches and viewpoints, some of them at first sight quite divergent, hint, perhaps, of some powerful synthesis. Although the contributions cover a rather vast spectrum, they are always focused on the general problematics of short-distance behavior in elementary particle theory.

The inductive aspects of the problem are treated with great rigor by Leutwyler and Otterson in an exhaustive review of the theoretical problems of deep inelastic scattering. The central role of the energy-momentum tensor is emphasized in the contribution by Wess, who presents a possible dynamical scheme, built in analogy to vector meson dominance for the currents. A central problem to the whole approach is faced by Schroer, who examines in particular the conformal invariance of the Gell-Mann/Low limit. The work by Ferrara, Gatto, Grillo, and Parisi summarizes the results of a concentrated effort to study the general consequences of conformal symmetry, mainly in a kinematical sense, by avoiding as much as possible additional dynamical assumptions. Very recent and powerful developments in conformal covariant quantum field theory are described by Mack. Conformal symmetry in Wilson's expansion is also discussed by Bonora, Ciccariello, Sartori, and Tonin.

The contribution by Bardeen, Fritzsch, and Gell-Mann is a remarkable and successful effort toward a final unification of the main concepts that have emerged in these last years in the description of elementary particles. Of such concepts Murray Gell-Mann has very often been the originator. The appealing prospect of possible contact with duality is dealt with by Del Giudice, Di Vecchia, Fubini, and Musto. Preparata's contribution reviews the concepts of light-cone dominance and the newly raised problem of crossing in the physics of the light cone. In Kleinert's work the problem of scaling is approached from the viewpoint of the infinite-component field formalism. The last contribution is from Kastrup, one of the pioneers of conformal symmetry, and addresses the problems of astrophysics, for which conformal symmetry suggests attractive speculations.

At the meeting in Frascati a most elegant introduction was given by Professor Drell, who reviewed his work with T. D. Lee on scaling and bound-state nucleons, and also some work by K. Johnson on quark description. The text of this paper was not submitted for the present volume since full accounts of these researches have in the meantime

appeared in print (S. D. Drell and T. D. Lee, *Phys. Rev. D* **5**, 1738, 1972; K. Johnson, *Phys. Rev. D* **6**, 1101, 1972).

Particular thanks are due to the numerous physicists who gave their enthusiastic and generous help to the Frascati meeting: to Professor C. Villi, President of the Italian National Institute of Physics; to Professor I. F. Quercia, Director of the Laboratories; to Dr. M. Greco and Dr. M. Ghigo-Ricci for their invaluable help in the organization; to Mr. S. Stipcich for his continuous assistance; to Professors W. Bardeen, G. Costa, J. Prentki, I. F. Quercia, L. Radicati, B. Renner, K. Symanzik, and B. Touschek, who acted as chairmen; and to the speakers and participants. While this book was being printed we learned of the death of our unforgettable friend Bruno Renner. Most of the physicists who have contributed to this volume knew him personally and were among his closest friends. I am sure all contributors will want to join me in dedicating this work to his memory.

I would like to commend the publishers for accepting a volume on a still-developing subject which I hope will provide us with a better understanding of fundamental interactions.

R. GATTO

Rome, Italy
February 1973

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CHAPTER 1

Theoretical Problems in Deep Inelastic Scattering

H. LEUTWYLER AND P. OTTERSON

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The first part (Sections 1–8) of this work is concerned with a number of *questions* concerning the analysis of inelastic electron scattering. (The discussion can easily be extended to weak processes and to nonforward matrix elements, but we stick to the prototype $e+p \rightarrow e+$anything because this process is best known experimentally.) The questions we want to discuss are the following:

1. Do the cross sections σ_L, σ_T determine the matrix element $\langle p|[j_\mu(x), j_\nu(0)]|p\rangle$ uniquely?

2. In particular, does the validity of the scaling laws imply canonical leading light-cone singularities?

26. A. O. Barut and H. Kleinert, *Phys. Rev.*, **156**, 1541 (1967).
27. H. Kleinert, *Lectures in Theoretical Physics*, Gordon and Breach, New York, 1968,
28. H. Kleinert, *Phys. Rev.*, **168**, 1827 (1968).
29. H. Kleinert, *Nucl. Phys.*, **B37** to be published.
30. H. Kleinert, *Phys. Rev.*, **163**, 1807 (1967).
31. K. Bardakci and G. Segré, *Phys. Rev.*, **159**, 1263 (1967); H. Leutwyler, *Acta Phys. Austriaca*, Suppl. V (1968) and *Springer Tracts in Modern Physics*, Vol. 50, p. 29, Springer-Verlag, Berlin, 1969; J. Jersak and J. Stern, *Nuovo Cimento*, **59**, 315 (1969); H. Leutwyler, J. R. Klauder, and L. Streit, *ibid.*, **66A**, 536 (1970); J. B. Kogut and D. E. Soper, *Phys. Rev. D*, **1**, 299 (1970); J. D. Bjorken, J. B. Kogut, and D. E. Soper, *ibid.*, **3**, 1382 (1971); J. M. Cornwall and R. Jackiw, *ibid.*, **4**, 367 (1971); A. Miklavc and C. H. Woo, University of Maryland preprint (August 1971); F. Jegerlehner, Bern preprint (February 1972).
32. H. Kleinert, *Proceedings of the XI Internationale Universitätswochen für Kernphysik, Schladming, Austria, 1972*, Chapter II.
33. S. Ferrara, R. Gatto, and A. F. Grillo, *Nucl. Phys.*, **B32**, 349 (1971).
34. B. Klaiber, *Lectures in Theoretical Physics*, Vol. XA, Gordon and Breach, New York, 1968; J. H. Lowenstein and B. Schroer, *Phys. Rev. D*, **3**, 1981 (1971).
35. K. Dietz and K. Meetz, Bonn preprint (1972).
36. H. Fritsch and M. Gell-Mann, talk presented at the International Conference on Duality and Symmetry, Tel-Aviv, Israel, 1971; see Chapter 7 of this volume.
37. M. Gomes and J. H. Lowenstein, University of Pittsburgh Preprint NYO-3829-90 (1972).

CHAPTER *11*

Some Astrophysical Speculations at Very High Energies

H. A. KASTRUP

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1. INTRODUCTION

Allow me a few unphysical remarks to begin with. When I became involved in the "conformal group" about 10 years ago (against the earnest advice of several people with tenure), I soon realized that it was looked at, among a large part of the physics community, almost like an "obscenity" a decent physicist just would not mention in public (how the situation was in private I don't know!). Time and other factors, however, have changed this too.

Gratifying as these social changes are, the physical results are not yet overwhelming. Despite several interesting and encouraging results in the realm of broken dilatation and conformal symmetries, many problems remain. As my contribution is intended to add some new problems, let us look first at some of the problems already known.

1. Experimentally we see "scaling" to an amazing degree in electron-nucleon scattering ("Bjorken scaling") and in hadron-nucleon collisions ("Feynman scaling"). However, the underlying dynamical frameworks and their relation to a scheme of asymptotic dilatation (or scale) and conformal invariance (operationally defined in coordinate space) are not very clear, despite interesting insights obtained in the last 1 or 2 years. After Wilson made his very important proposal[1] as to how to put a notion of asymptotic dilatation invariance into an operational form by making operator expansions in the neighborhood of vanishing Euclidean distances of two field quantities, Gatto and his collaborators discussed[2] the relation of these expansions to current algebra and Bjorken scaling. In an interesting paper[3] Mack analyzed this connection without using equal-time commutators and discussed Bjorken scaling for the inelastic electromagnetic nucleon structure functions from Wilson's expansion, positivity, and some smoothness assumptions (unsubtracted dispersion relation). Both positivity and smoothness assumptions play a central role, and for this reason there seems to be no one-to-one relation between dilatation invariance, as incorporated into Wilson's expansions, and Bjorken scaling.

If one requires asymptotic operator expansions not only for vanishing Euclidean but also for Minkowski distances, one gets the light-cone expansions as proposed by several people.[4] If one *assumes* the light-cone singularities of the relevant current-current matrix element to be dominant and to be canonical, then scaling á la Bjorken follows. However, I do not know[5] any convincing argument that the Bjorken limit in momentum space should be determined by the singularities on the light cone and nothing else! It seems that the postulates of canonical light-cone dominance, on the one side, and Wilson's expansions plus smoothness conditions as discussed by Mack, on the other side, are somehow equivalent, but no exact mathematical statements about this exist.

All this shows that the relation between Bjorken scaling in inelastic electron-nucleon scattering and dilatation invariance at short distances or on the light cone is not simple, at least at the present state of our insights. The situation becomes even more intricate (or strange!) if one applies[6] conformal invariance to the light-cone expansions by requiring the operators on the right-hand and left-hand sides of the expansions to be irreducible tensor operators with respect to the special conformal generators. The implication[6] that all the operators on the right-hand side with their higher and higher spins have to be conserved currents is surprising indeed, and the meaning of this is not yet clear.

Not yet understood at all is the relation (if any!) between Feynman scaling in purely hadronic interactions and approximate dilatation (conformal) invariance formulated in coordinate space, for instance, in the sense that the degree of homogeneity of certain space-time singularities of the quantity[7]

$$\langle p_1 p_2 | j(x) j(0) | p_1 p_2 \rangle$$

is determined by an appropriate operational definition of asymptotic dilatation invariance in a way perhaps similar to that in the case of inelastic electron-nucleon scattering. I shall come back to this point at the end of the chapter.

Despite their importance, the operator expansions do not contain very much dynamical information. I shall mention two recent proposals to bring more dynamical properties into a scheme of asymptotic dilatation and (or) conformal invariance.

2. Fritsch and Gell-Mann proposed[8] an extension of the light-cone expansions by abstracting algebraic closure relations from the free quark-gluon model, with canonical dimensions on the light cone even for interacting fields. This field is presently under intense investigation; and as soon as the high-energy neutrino beams in Batavia are operating, we shall know more about the validity of some of the sum rules derived in the framework of that scheme.

3. A very interesting proposal[9,10] for combining dilatation and conformal invariance in the dynamical framework of Green's functions is due to Polyakov and Migdal, followed by an extensive analysis[11–14] by Mack and Todorov, Mack and Symanzik, Schroer, and others. The crucial observation of Polyakov (for nonrelativistic correlation functions) was that conformal invariance, in addition to dilatation invariance, determines not only the two-point function but also the three-point function up to very few parameters. When the higher n-point functions are built up by skeleton expansions, the free parameters are in principle determined by integral (Dyson-Schwinger) equations. The big problem is to find nontrivial solutions of these equations. The present status of this very interesting field has been summarized by Mack.[15]

Again, the relation of this dynamical approach to physical processes still has to be unambiguously established and analyzed; for instance, how the operator expansions fit in, and whether one can derive Bjorken scaling, must be determined.

4. From all that I have said above it is clear that we need more comparisons with experiments in order to see in what way (if at all) nature makes use of some kind of asymptotic dilatation and conformal invariance

at very high energies, by starting from a suitable operational definition of these symmetries in coordinate space. This is one reason why I want to discuss an additional attempt to make contact between the real world and some more or less reasonable hypothesis about the way in which dilatation and conformal invariance may show up as an approximate symmetry at very high energies. The main idea is as follows.[16]

Suppose that some reactions not only show scaling at very high energies but also conserve, at least approximately, dilatation and conformal momenta, perhaps in the weak sense that in some quasi-elastic scattering reactions the expectation values of these momenta are approximately the same before and after those collisions. Such an assumption is similar to that of Nambu and Lurie[17] about the weak conservation of chirality in certain reactions.

Suppose further that we have a highly relativistic gas of particles reacting with each other in the sense just described, that is, conserving dilatation and conformal momenta, at least in a reasonable approximation. If such a very hot gas is in equilibrium, we expect equilibrium distribution functions to exist[18, 19] which are solutions of the relativistic Boltzmann equation and the logarithms of which are linearly related to the quantities conserved in those binary or "quasi"-binary collisions.

The most important example of such a gas might have been the universe right after the big bang. For this reason the following discussion may be of some physical, as well as mathematical, interest.

In order to separate the purely mathematical part from the much more difficult physical one, I shall discuss first the most important properties of a highly relativistic gas in equilibrium such that its binary collisions conserve dilatation or conformal momenta.

Afterwards I shall try to make contact with some known properties of the universe. The most difficult task is the analysis of several known high-energy processes which show scaling and which might have led, at least approximately, to distribution functions of the type mentioned above. If such distributions were present right after the big bang, then they might have had a very important effect on the evolution of the universe, as we shall see.

2. PROPERTIES OF DISTRIBUTION FUNCTION FOR SYSTEMS WITH DILATATION OR CONFORMALLY INVARIANT BINARY COLLISIONS

In this part we can use the results of Ehlers, Geren, and Sachs,[20] who extended previous work by Tauber and Weinberg.[20] We merely have to

apply these results to the flat Minkowski space.[21] Accordingly, a locally isotropic distribution function $f(x,p)$ for particles with momentum p and vanishing or negligible rest masses has the general form

$$f(x,p)=g[\xi^{\mu}(x)p_{\mu}]$$

where g is a differentiable function, and $\xi^{\mu}(x)$ a conformal Killing vector, that is, it has the property

$$\partial_{\nu}\xi_{\mu}+\partial_{\mu}\xi_{\nu}=g_{\mu\nu}\lambda(x)$$

$\lambda(x)$ being a scalar function. The Killing vector defines the basic velocity field $u^{\mu}(x)$ by

$$\xi^{\mu}(x)=\alpha(x)u^{\mu}(x), \qquad \xi^{\mu}\xi_{\mu}=\alpha^2$$

In order to simplify our discussion technically, we shall deal only with Boltzmann distributions here; the generalizations to Bose-Einstein and Fermi-Dirac statistics can be obtained by using, for example, the results of ref. 18.

a. *Dilatations*

The transformations

$$D(\alpha): x^{\mu}\rightarrow\hat{x}^{\mu}=e^{\alpha}x^{\mu}, \qquad \alpha \text{ a real constant}$$

give the conformal Killing vector

$$\xi_D{}^{\mu}(x)=\left(\frac{\partial\hat{x}^{\mu}}{\partial\alpha}\right)_{\alpha=0}=x^{\mu}$$

and the corresponding equilibrium distribution function is

$$f(x,p)=F_D\exp\left[-\frac{p^{\mu}u_{\mu}(x)}{kT(x)}\right] \tag{1}$$

where

$$u^{\mu}(x)=\frac{x^{\mu}}{(x^2)^{1/2}}, \quad x^2>0, \quad kT(x)=\frac{1}{\beta_D(x^2)^{1/2}}, \quad \text{and } F_D,\beta_D=\text{const}$$

The invariant particle density $n(x)$ is defined by the current

$$n^{\mu}(x)=\int\frac{d^3p}{p^0}p^{\mu}f(x,p)=n(x)u^{\mu}(x)$$

and has the value

$$n_D(x)=8\pi F_D[kT(x)]^3$$

The energy-momentum tensor $T_{\mu\nu}(x)$ is defined by

$$T_{\mu\nu}(x)=\int\frac{d^3p}{p^0}p_{\mu}p_{\nu}f(x,p)=(\hat{\mu}+\hat{p})u_{\mu}u_{\nu}-g_{\mu\nu}\hat{p}$$

where $\hat{\mu}(x)$ is the proper energy density, and $\hat{p}(x)$ is the proper pressure. Because of the negligible rest masses the trace $T_{\mu}{}^{\mu}$ vanishes, and we have $3p=\hat{\mu}$,

$$\hat{p}_D(x)=8\pi F_D[kT(x)]^4$$

Because $u_{\mu}=\partial_{\mu}(x^2)^{1/2}$, the flow associated with the velocity field is hypersurface orthogonal. In addition we have

$$u_{\mu;\nu}\equiv\partial_{\nu}u_{\mu}(x)=\left(\frac{1}{x^2}\right)^{1/2}(g_{\mu\nu}-u_{\mu}u_{\nu}) \qquad (2)$$

that is, the expansion velocity θ has the value $3/(x^2)^{1/2}$. The acceleration $\dot{u}_{\mu}=u_{\mu;\nu}u^{\nu}$ vanishes.

Transformation to co-moving coordinates is obtained by the substitution

$$x^0=\tau\cosh w, \qquad r=\tau\sinh w, \qquad \theta\to\theta, \qquad \varphi\to\varphi$$
$$\tau=(x^2)^{1/2}, \qquad ds^2=d\tau^2-\tau^2[dw^2+\sinh^2 w(d\theta^2+\sin^2\theta\, d\varphi^2)] \qquad (3)$$

In the co-moving system the velocity field is given by $\bar{u}^{\mu}=g_0{}^{\mu}$.

We see that the system so obtained is just Milne's universe,[22] which has been discussed extensively in the literature,[23] so that we do not need to repeat those discussions here. What is remarkable is that we obtain the system by the *dynamical* requirement that the binary collisions be invariant under dilatations.

Let us illustrate this dynamical background of the expansion by the following example. The scattering of an ultrarelativist particle off a potential $1/r$ conserves[24] the dilatation momentum $D=x^{\mu}p_{\mu}$. If we rewrite this as $\mathbf{x}\cdot\mathbf{p}=p_0x^0-D$, we see that for constant D the virial $\mathbf{x}\cdot\mathbf{p}$ increases linearly with time, that is, the motions of the particles in the system cannot be bounded!

Up to now we have taken into account dilatation invariance only. In reality we always have energy-momentum conservation. If we incorporate energy conservation, we obtain the same formulas as above, but temperature and velocity field now take the forms

$$kT(x)=(\beta_t{}^2+2\beta_t\beta_Dx^0+\beta_D{}^2x^2)^{-1/2}$$
$$(u^0(x),\mathbf{u}(x))=kT(x)(\beta_t+\beta_Dx^0,\beta_D\mathbf{x})$$

This means that we have two parameters instead of one determining the temperature and other properties, where the new one, β_t (the index t refers to time translation) is just the inverse temperature at $x=0$! It vanishes only if $T(x=0)=\infty$.

b. *Special Conformal Transformations*

The special conformal (SC) transformations

$$\mathrm{SC}(c):x^{\mu}\to\hat{x}^{\mu}=\frac{x^{\mu}-c^{\mu}x^2}{\sigma(x;c)}, \qquad \sigma(x;c)=1-2c\cdot x+c^2x^2$$

give the four conformal Killing vectors:

$$\xi_{\alpha}=(\xi_{\alpha}^{\mu}), \qquad \xi_{\alpha}^{\mu}=2x^{\mu}x_{\alpha}-g_{\alpha}^{\mu}x^2, \qquad \alpha,\mu=0,1,2,3$$

We choose a constant timelike unit vector $b, b^2=1$, and define the timelike vector $\xi_c^{\mu}=\xi_{\alpha}^{\mu}b^{\alpha}$, $\xi_c{}^2=(x^2)^2$. In this case the equilibrium distribution function, velocity, and temperature fields are given by

$$f(x,p)=F_c\exp\left(\frac{-u^{\mu}p_{\mu}}{kT(x)}\right)$$
$$u^{\mu}(x)=-x^2\partial^{\mu}\left(\frac{b\cdot x}{x^2}\right), \qquad kT(x)=\frac{1}{\beta_cx^2}, \qquad F_c,\beta_c=\text{const.} \qquad (4)$$

From this we get for the invariant density and the invariant pressure:

$$n_c(x)=8\pi F_c(kT)^3, \qquad \hat{p}_c(x)=8\pi F_c(kT)^4$$

Differentiation of the velocity field gives

$$u_{\mu;\nu}(x)=\dot{u}_\mu u_\nu+\left(\frac{2x\cdot b}{x^2}\right)(g_{\mu\nu}-u_\mu u_\nu)$$
$$\dot{u}_\mu=u_{\mu;\nu}u^\nu=\left(\frac{2}{x^2}\right)(x\cdot bu_\mu-x_\mu) \tag{5}$$

that is, now the acceleration does not vanish! With $b=(1,0,0,0)$ we get $u_\mu=-x^2\partial_\mu(x^0/x^2)$, which shows the flow to be hypersurface orthogonal. Transformation to co-moving coordinates is obtained by the mapping

$$\tau=\frac{x^0}{x^2},\quad \rho=\frac{r}{x^2},\quad \theta\to\theta,\quad \varphi\to\varphi,\quad u_\mu\to\bar{u}_\mu=(\tau^2-\rho^2)g_{\mu 0}$$

$$ds^2=(\tau^2-\rho^2)^{-2}[d\tau^2-d\rho^2-\rho^2(d\theta^2+\sin^2\theta\, d\varphi^2)]$$

From the expression for the line element we see that the system is isotropic, but not homogeneous. This means that we do not have a Robertson-Walker metric and no Friedman universe!

For our astrophysical discussions below it is important to have red-shift-distance relations. Suppose that light having wavelength λ and emitted at $(t,\mathbf{x})\leftrightarrow(\tau,\rho)$ arrives at $\mathbf{x}=0$ at the time $t_0=t+r/c$ with wavelength λ_0. Then one obtains in the usual manner for the red shift $z=(\lambda_0-\lambda)/\lambda$ and those distances the following relations:

$$\rho=\left(\frac{\tau_0}{2}\right)z,\qquad \tau=\left(\frac{\tau_0}{2}\right)(z+2) \tag{6}$$

$$r=\frac{1}{2}\frac{z}{1+z}t_0,\qquad t=\frac{1}{2}\frac{2+z}{1+z}t_0,\qquad \frac{1}{x^2}=\left(\frac{1}{t_0}\right)^2(1+z) \tag{7}$$

For $z\ll 1$ the first of Eqs. 7 reduces to Hubble's law:

$$r=\left(\frac{1}{H_0}\right)z,\qquad H_0=\frac{2}{t_0} \tag{8}$$

For larger red shifts, however, Hubble's law is not a good approximation and would give distances larger than the actual ones.

At this point two comments are in order.

1. For $r^2\ll t^2$, that is, "locally," the system is isotropic *and* homogeneous, and Hubble's law (Eq. 8) is valid. The corresponding local Robertson-Walker metric has an expansion function $R=R_1t^2$, $R_1=\text{const.}$; this is not a solution of Friedman's equation,[23] but one can show that it is a solution of the corresponding Brans-Dicke equation[25] with $\omega=-\frac{3}{2}$ (see ref. 26), vanishing spatial curvature, and the Brans-Dicke scalar:

$$\frac{1}{\varphi(t)}=\left(\frac{1}{\varphi_1}\right)t^6,\qquad \varphi_1=\frac{2^8\pi^2F_c}{g\beta_c^{\,4}}$$

2. Incorporating energy conservation of the binary collisions gives, in the same manner as in the case of the dilatations above, the following modifications:

$$kT(x)=\left[\beta_t^{\,2}+2\beta_t\beta_c(t^2+r^2)+\beta_c^{\,2}(x^2)^2\right]^{-1/2}$$

$$(u_0(x),\mathbf{u}(x))=kT(x)(\beta_t+\beta_c(t^2+r^2),2t\beta_c\mathbf{x})$$

The parameter β_t again is the inverse temperature for $x=0$. For very large times it becomes negligible.

3. SOME VERY PRELIMINARY ASTROPHYSICAL REMARKS

It is obviously tempting to compare the systems that we have been discussing with our expanding universe. Because the first one is the well-known model of Milne, I shall confine myself to the second system, the "conformal" one.

Locally, that is, for $r^2\ll t^2$, the factor 2 in Hubble's constant $H_0=2/t_0$ is remarkable. If taken seriously, the factor 2 in $t_0=2/H_0$ would eliminate an "age" problem,[27] if there is one, and if there is any relation at all of our system to the realities surrounding us! Even if our system had something to do with the properties of the universe right after the big bang, the question remains whether there would be still traces around.

Furthermore, we have a strong decrease of the density $n_c(t)$ with time t because $n_c(t)t^6=\text{const}$. This has to be compared with the much weaker gravitational decrease in, for instance, the Einstein-De Sitter model,[23] where $n(t)t^2=\text{const}$.

In our neighborhood traces of our distribution in the form of neutrinos, photons, or fast cosmic rays may still be around. Something amusing is the following. The motion of very fast particles in the gravitational potentials $1/r$ conserves not only energy, angular momentum, and dilatation momentum, but also the time component of the conformal momenta. Thus the motion (scattering) of very fast cosmic rays in the gravitational centers of relatively slow masses does not destroy the "conformal" properties of a gas of fast particles. Somehow the potentials $1/r$ act similarly to the energy-conserving walls of a box filled with a gas.

If the "conformal" distribution was of some importance right after the big bang, the more traces would probably be found the farther one goes back in time, or the larger the red shifts z are. Now we have seen that, for red shifts which are not small compared to 1, we have a considerable modification of Hubble's law, as well as strong deviations from homogeneity. I am not able to propose a detailed mechanism, but it is, of course, tempting to associate some of the remarkable properties of quasars with our system, in the sense that the remnants of the "primeval" conformal distribution are somehow coupled to quasars. More work is certainly needed to clarify this point and to go beyond uncertain speculations!

4. HIGH-ENERGY PROCESSES WHICH MIGHT HAVE LED TO CONFORMAL DISTRIBUTION FUNCTIONS

A very important problem in the context of our considerations is the following: Which high-energy processes that we know about might give rise, at least in a reasonable approximation, to the "dilatational" and conformal distribution functions discussed above? Although I shall not be able to give a convincing answer to this question, there are a number of theoretical and experimental indications that those systems should be taken seriously.

In general, in this section I shall discuss several reactions with cross sections of a form which can be related to an effective dilatation and conformally invariant "potential," such as the Coulomb potential, or a Born approximation in kinematical regions where all rest masses are negligible and where there could be approximate conservation of dilatation and conformal momenta.

a. *Purely Electromagnetic Interactions in Lowest Order*

(1) Compton scattering off electrons for $s, |t| \gg m_e^2$:

$$\frac{d\sigma}{d|t|} = \frac{2\pi\alpha^2}{s^2}\left(\frac{s}{|t|} + \frac{|t|}{s}\right)$$

$$s = (p_e + p_\gamma)^2, \qquad t = (p_e - p_e')^2, \qquad p_e' = \text{momentum after scattering}$$

(2) Electron-electron scattering for $s, |t| \gg m_e^2$:

$$\frac{d\sigma}{d|t|} = \frac{2\pi\alpha^2}{t^2 s^2}(t^2 + 2st + 2s^2)$$

The above two cross sections can be obtained from the dilatation and conformally invariant Lagrangian in quantum electrodynamics without the electron rest mass term. Higher orders, which will have to include that mass term, give nonscaling corrections, but we have at least approximate scaling.[24]

b. *Inclusive Semihadronic Processes*

Not only does the reaction $e + N \to e' + \text{anything}$ show scaling of the structure functions, but also the differential total cross section is Coulomb-like:[28, 29]

$$\frac{d\sigma_{\text{tot}}}{d|t|} \approx \frac{4\pi\alpha^2}{t^2}.$$

This process may be looked at, at very high energies, as an effective binary electron scattering off a Coulomb potential. However, in such a case we have conservation of the dilatation momentum[24] and of the time component of the conformal momenta, where that component has the form

$$K^0 = 2x^0(x \cdot p) - p^0 x^2 - 2gr, \qquad g = \text{coupling const.}$$

We see that in this case we have some kind of quasi-binary collision, which we need in order to obtain the distribution functions of interest to us.

In weak interactions we can apply the same reasoning only[30] if there is an intermediate boson W and if the momentum transfer of the leptons is larger than its mass m_W.

c. *Purely Hadronic Interactions*

In inclusive hadronic collisions $a + b \to c + \text{anything}$, "Feynman" scaling is observed[31] at very large energies. In other words, the cross section for the particle c with longitudinal center of mass momentum $p_\parallel$ and transverse momentum $p_\perp$ is to a good approximation of the form

$$\frac{d^2\sigma}{p_\perp dp_\perp dy} = f(p_\perp, y), \qquad y = \frac{2p_\parallel}{\sqrt{s}}$$

Either if the function $f(p_\perp,y)$ factorizes, $f(p_\perp,y)=f_1(p_\perp)f_2(y)$, or if we integrate over $p_\perp$, we get

$$\frac{d\sigma}{dy}=g(y)$$

This form of the production cross section may be interpreted as follows. A cross section in *one* space dimension, defined by the decrease $\Delta j=-\sigma jn(x^3)\Delta x^3$ of the current density $j(x^3)$ along Δx^3, is dimensionless. This implies that, if the dynamics do not contain any fixed length like rest masses or coupling constants with nonvanishing dimension of length and so forth, the production cross section for one particle in the final state, after summing over all other secondaries, can only be a function of the ratio y. In this sense purely hadronic interactions exhibit one-dimensional (longitudinal) dilatation invariance.[24] A formal field theoretical example for such a situation provides the Thirring model. The example is formal because there seems to be no nontrivial S-matrix.

We have seen that there are a number of important high-energy reactions which might have led, right after the big bang, to approximate distributions for the primeval matter which may have been at least partially of the type I have discussed above. This is, of course, a rather weak statement, its weakness having various obvious sources, for some of which I am fully responsible, for others not so much! But I hope I have indicated that this field is worth investigating in the context of approximate and broken conformal symmetries.

REFERENCES AND NOTES

1. K. G. Wilson, *Phys. Rev.*, **179**, 1499 (1969).
2. S. Ciccariello, R. Gatto, G. Sartori, and M. Tonin, *Ann. Phys. (N.Y.)*, **65**, 265 (1971).
3. G. Mack, *Nucl. Phys.*, **B35**, 592 (1971).
4. Y. Frishman, *Ann. Phys. (N.Y.)*, **66**, 373 (1971); R. Brandt and G. Preparata, *Nucl. Phys.*, **B27**, 541 (1971).
5. H. Leutwyler and P. Otterson, Chapter 1 in this volume.
6. S. Ferrara, R. Gatto, A. F. Grillo, and G. Parisi, *Phys. Letters*, **38B**, 333 (1972).
7. A. H. Mueller, *Phys. Rev. D*, **2**, 2963 (1970).
8. H. Fritsch and M. Gell-Mann, "Proceedings of the Coral Gables Conference on Fundamental Interactions at High Energy," University of Miami, 1971; Chapter 7 of this volume.
9. A. M. Polyakov, *Zh. Eksperim. i Teor. Fiz. Pis. Red.*, **12**, 7538 (1970); transl. *JETP Letters*, **12**, 7381 (1970).
10. A. A. Migdal, *Phys. Letters*, **37B**, 98, 386 (1971).

11. G. Mack and I. T. Todorov, Triest Preprint IC/71/139; Chapter 5 in this volume.
12. G. Mack and K. Symanzik, Desy Preprint 72/20; Chapter 5 in this volume.
13. B. Schroer, Chapter 3 in this volume.
14. G. Parisi and L. Peliti, *Lettere Nuovo Cimento*, **2**, 627 (1971).
15. G. Mack, Chapter 5 in this volume.
16. H. A. Kastrup, *Phys. Letters*, **37B**, 521 (1971).
17. Y. Nambu and D. Lurie, *Phys. Rev.*, **125**, 1429 (1962).
18. J. Ehlers, "General Relativity and Cosmology," in *Proceedings of the International School of Physics*, "Enrico Fermi," Academic Press, New York, 1971.
19. J. M. Stuart, *Lecture Notes in Physics*, Vol. 10, Springer-Verlag, Berlin, 1971.
20. J. Ehlers, P. Geren, and R. K. Sachs, *J. Math. Phys.*, **9** 1344 (1968); G. E. Tauber and J. W. Weinberg, *Phys. Rev.*, **122**, 1342 (1961).
21. We use the metric $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$, all others vanish; we further have $\hbar=1=c$ and write ∂_μ for $\partial/\partial x^\mu$.
22. E. A. Milne, *Kinematic Relativity*, Oxford University Press, 1948.
23. H. P. Robertson and T. W. Noonen, *Relativity and Cosmology*, Saunders, Philadelphia, 1968; W. Rindler, *Essential Relativity*, Van Nostrand-Reinhold, New York, 1969.
24. H. A. Kastrup, "Habilitation," thesis, Munich, 1964; the larger part of this thesis is contained in *Nucl. Phys.*, **58**, 561 (1964).
25. C. Brans and R. H. Dicke, *Phys. Rev.*, **124**, 925 (1961).
26. J. L. Anderson, *Phys. Rev. D*, **3**, 1689 (1971).
27. A. R. Sandage, *Phys. Today*, February 1970, p. 34.
28. H. A. Kastrup, *Phys. Rev.*, **147**, 1130 (1966); *Schladming Lectures in Particles, Currents, Symmetries*, by P. Urban, Springer-Verlag, Vienna, 1968, p. 407.
29. J. D. Bjorken, *Proceedings of the International School of Physics*, "Enrico Fermi," Academic Press, New York, 1968.
30. H. A. Kastrup, *Nuovo Cimento*, **48A**, 271 (1967).
31. Oxford Conference on High Energy Collisions, April 1972; proceedings to be published.