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ON THE PROBLEM OF CAUSALITY IN CONFORMAL INVARIANT THEORIES

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This is a short review on the considerable progress made in the last year or so in the field of conformal invariant theories, especially as far as the problem of causality is concerned.

I. STATEMENT OF THE PROBLEM

Let $M^4 = \{E\}$ be the Minkowski space of events E with coordinates $x^j(E)$, $j = 0,1,2,3$, and the metric form

$$g(\ x(2),x(1)\ ) = (x^\circ(2)-x^\circ(1))^2 - (\vec{x}(2)-\vec{x}(1))^2$$
$$= (x(2)-x(1))^2;\ x(i)=x(E_i),i=1,2$$

A causal ordering of two timelike events: $E_2>E_1$ ("E_2 later than E_1") is defined by the functions

$$g(x(2),x(1)) > 0,\ \mathrm{sign}(x^\circ(2)-x^\circ(1)) > 0 \qquad (1)$$

This notion of causal ordering is invariant under the orthochronous Poincaré group

$$P_{10}(a,\Lambda):\ x^j \to \hat{x}^j = \Lambda^j{}_k\, x^k + a^j \quad ,$$

(the index "10" denotes the number of independent parameters) and the dilations

$$D_1(\rho): \hat{x}^j \to x^j = \rho x^j, \ \rho>0.$$

Under very general assumptions it can be shown[1] that these transformations of M^4 onto M^4 are also the most general ones which leave the causal ordering (1) invariant.

Quantized fields F(x) are operator-valued distributions and transform like

$$U(a,\Lambda)F(x)U^{-1}(a,\Lambda) = S(\Lambda^{-1})F(\Lambda x+a),$$

where $S(\Lambda^{-1})$ is a finite matrix, depending on the spin of F. Systems which are also dilation-invariant:

$$U(\rho)F(x)U^{-1}(\rho) = \rho^{d_F} F(\rho x),$$
$$d_F\text{: dimension of F},$$

are much more restricted, because $U(\rho)P^2U^{-1}(\rho) = \rho^{-2}P^2$, i.e. either $P^2=0$ or the mass spectrum is continuous!

Problems arise in connection with the "special conformal" transformations of M^4.

$$SC_4(c): x^j \to \hat{x}^j = (x^j - c^j x^2)/\sigma(x;c) \ ,$$
$$\sigma(x;c) = 1-2c\cdot x+c^2x^2$$

As σ (x;c) is not positive definite, the mapping (2) is illdefined for $\sigma = 0$. In addition we have

$$x^2 \to \hat{x}^2 = x^2/\sigma(x;c) \ .$$

which means $\hat{x}^2 < 0$ for $x^2 > 0$, $\sigma < 0$, i.e. the group SC_4 can change the geometrical (global) causal ordering (1) of two events! Because of

$$dx^j dx_j \rightarrow (\Lambda/\sigma^2)\, dx^j dx_j$$

it leaves invariant a <u>local</u> causal ordering, however.

As conformal invariant theories appear to have interesting properties[2], it seems to be worthwhile to look for remedies for these diseases.

It should be mentioned that in the case $P^2=0$ (free mass zero particles) and arbitrary helicities it is possible[3] to have a well-defined field theory on M^4 even for <u>finite</u> conformal mappings, the reason being the special transformation properties of test functions under $SC_4(c)$.

II. SOLUTION OF THE GEOMETRICAL PROBLEM

The first step is, to extend the Minkowski space in a minimal way, so that the action of SC_4 or of the 15-parameter group $C_{15}(a,\Lambda,\rho,c)$ is well-defined on the new manifold.[4] This is being achieved by introducing projective coordinates such that

$$x^j = \frac{\eta_j}{\eta^4 + \eta^5},\quad x^j \varepsilon M^4,\quad \eta^4 + \eta^5 \neq 0.$$

The coordinates η^μ, $\mu=0,\dots,5$, are restricted by the relation

$$\eta^\mu . \eta_\mu \equiv (\eta^0)^2 - (\eta^1)^2 - \dots - (\eta^4)^2 + (\eta^5)^2 = 0, \eta \neq 0.$$

The points E of the extended Minkowski space, called M_c^4, are in one-to-one correspondence to the rest classes

$$[\eta] = \{\eta, \eta(2) = \lambda\eta(1),\ \lambda \neq 0\}\ .$$

Taking special representatives of these classes:

$$(\eta^1)^2 + \dots + (\eta^4)^2 = \Lambda = (\eta^0)^2 + (\eta^5)^2,$$

we see that

$$M_c^4 \sim (S^1 \times S^3)/Z_2,$$

where S^n is the n-dimensional unit sphere and Z_2 means identification of opposite points, S^1 being the compactification of the time line R^1 and S^3 the compactification of R^3. (M_c^4 being compact cannot be described by a single coordinate system, actually one needs at least four[4])!)

If we define $x = \eta^4 + \eta^5$, we have on M^4

$$\text{sgn}\,[x^o(2)-x^o(1)] = \text{sgn}\,[(\eta^o(2)\eta^5(1)-\eta^5(1)\eta^o(2)/x(1)x(2)]$$

$$\text{for} \quad (x(2)-x(1))^2 = -2(\eta^\mu(2)\eta_\mu(1))/x(1)x(2) > 0, \qquad (3)$$

$$\text{and} \qquad dx^j\, dx_j = (1/x^2)\,d\eta^\mu\, d\eta_\mu .$$

The twofold covering group $O^\uparrow(2,4)$ of $C_{15}^\uparrow$ acts linearly on the coordinates η^μ:

$$\eta_\mu \to \hat\eta^\mu = w^\mu{}_\nu \eta^\nu, \quad W^T g W = g, \quad W = (w^\mu{}_\nu), \quad \text{diag.}\,(g) = (1,-1,-1,-1,-1,1).$$

The orthochronous part of $O(2,4)$ is defined by $\text{sgn}(w^o{}_o w^5{}_5 - w^o{}_5 w^5{}_o) > 0$. One can show[4] that it leaves the function

$$\text{sgn}[\eta^o(2)\eta^5(1) - \eta^o(1)\eta^5(2)]$$

$$\text{invariant if } \eta^\mu(2)\,\eta_\mu(1) = 0.$$

In connection with Eqs. (3) this gives us a C_{15}-invariant local causal ordering on M_c^4, but not a global one. The impossibility of a global causal ordering on M_c^4 follows also immediately from the topological structure $S^1 \times S^3$: there is no global ordering of points on S^1! The way out globally is the following[5)-8)]: The manifold M_c^4 is not simply connected. Its universal covering space is $\tilde{M}^4 \sim R \times S^3$,

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the points $\tilde{E}$ of which can be described by the coordinates

$$\tilde{x}(\tilde{E}) = (\tau \in R,\ \underset{\sim}{n} = (n^1,\ldots,n^4;\ \underset{\sim}{n}^2 = 1)).$$

The projection $\tilde{M}^4 \to M^4_c$ is given by

$$\eta^o = \lambda \sin\tau,\ \eta^5 = \lambda\cos\tau,\ \eta^\mu = \lambda\eta^\mu,\ \mu=1,\ldots,4;\ \lambda\neq 0$$

From this one sees immediately the infinitely sheeted covering of M^4_c. On $M^4 \subset M^4_c$ we have

$$x^o = \frac{\sin\tau}{\cos\tau + n4},\quad \vec{x} = \frac{\vec{n}}{\cos\tau + n4},$$

$$-\frac{1}{2}\pi < \tau < \frac{1}{2}\pi,\ n4 \geq 0.$$

The metric on $\tilde{M}^4$ is given by

$$d\tau^2 - (d\underset{\sim}{n})^2 = \frac{1}{(\eta^o)^2 + (\eta^5)^2} \qquad (4)$$

$$d\eta^\mu d\eta^\mu,\ \underset{\sim}{n}\,d\underset{\sim}{n} = 0,\ \eta^\mu d\eta_\mu = 0.$$

or, globally,

$$(\tau_2-\tau_1)^2 - \text{across}\ (\underset{\sim}{n}(2)\cdot\underset{\sim}{n}(1)).$$

A global causal ordering, $\tilde{E}_2 > \tilde{E}_1$, can now be defined for time-like events by

$$\tau_2-\tau_1 > 0,\ (\tau_2-\tau_1)^2 - \text{arcoss}\ (\underset{\sim}{n}(2)\underset{\sim}{n}(1)) > 0. \qquad (5)$$

Projected on M^4 this definition coincides with that of Eq. (1)!

It remains the problem, whether the notion (5) is invariant under the groups discussed above. For this purpose we have to deal with the universal covering group of $C_{15}^{\uparrow}$: the group SU(2,2) is a 2-fold covering of $SO^{\uparrow}(2,4)$ and a 4-fold covering of $C_{15}^{\uparrow}$. It has the topological (not group theoretical!) structure

$$SU(2,2) \sim U(1) \times SU(2) \times SU(2) \times R^8,$$

implying[9] the structure

$$\widetilde{SU(2,2)} \sim R \times SU(2) \times SU(2) \times R^8$$

for the universal covering group. The action of SU(2,2) on $\tilde{M}^4$ can be computed explicitly[7] and it can be shown to leave the ordering (5) invariant!

Two further remarks:

1. The center $\tilde{Z}$ of $\widetilde{SU(2,2)}$ is an infinite, discrete abelian group, isomorphic to $Z_\infty \times Z_2$, where Z_2 corresponds to the center of one of the SU(2)-groups and distinguishes between fermions and bosons. The other factor Z_∞ is given by

$$Z_\infty = \{z^n, n=0, \pm 1, \pm 2, \ldots, z = R_{05}(\pi)\Pi\},$$
$$R_{05}(\pi): \tau \to \tau + \pi, \quad \Pi: \underset{\sim}{n} \to -\underset{\sim}{n}$$

The center $\tilde{Z}$ seems to play an important role <u>dynamically</u> (see the following paragraph)!

2. The group of motions for the metric $d\tau^2 - (dn)^2$ is the group $T(\sigma) \times O(4)$, where $T(\sigma)$: $\tau \to \tau + \sigma$.

III. PROPERTIES OF FIELDS

Let $\tilde{g} \varepsilon \widetilde{SU(2,2)}$, $U(\tilde{g})$ a unitary representation of $\widetilde{SU(2,2)}$ and $A(\tilde{x})$ a scalar field[10] on $\tilde{M}^4$:

$$U(\tilde{g})\, A(\tilde{x})\, U(\tilde{g})^{-1} = A(\tilde{g}\tilde{x})$$

If locality holds on $\tilde{M}^4$:

$$[A(\tilde{x}(2)), A(\tilde{x}(1))] = 0, \quad \tilde{x}(2), \tilde{x}(1) \; \varepsilon \; \tilde{M}^4 \text{ and relatively spacelike,}$$

then it holds for arbitrary spacelike points on $\tilde{M}^4$, because SU(2,2) acts transitively on pairs of relatively spacelike points:

$$0 = U(\tilde{g})\ [A(\tilde{x}(2)),\ A(\tilde{x}(1))]\ U^{-1}(\tilde{g}) = [A(\tilde{g}\tilde{x}(2)),A(\tilde{g}\tilde{x}(1))].$$

Conformal invariant field theories, especially in the Euclidean region, have been studied intensely by Mack and others[2]. An interesting physical problem is the following[11,12,2]: In which way do fields $A(\tilde{x})$ on $\tilde{M}^4$ differ "over" ("under") the same points $x \varepsilon M^4$? Examples show that they differ in an essential way (i.e. not just by a phase), *if* the dimension of the field has an anomalous (dynamical!) part! This can best be seen by looking at the transformation of $A(\tilde{x})$ under the center $\tilde{Z}$, because $\tilde{Z}$ takes a sheet of $\tilde{M}^4$ to the next one, but acts as the identity on M^4.

As the fields with anomalous dimensions are *not* observables, their behavior under $\tilde{Z}$ seems to be acceptable. On the other hand, *observable* quantities like currents do not have anomalous dimensions and, therefore, do not appear to differ in an essential way in different sheets over the same point E of M^4. This means that the physics in different sheets is the same, as one would expect it to be! However, this problem needs further investigation!

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