PROPERTIES OF DILATATIONS AND CONFORMAL TRANSFORMATIONS IN MINKOWSKI SPACE*

T. H. Go, H. A. Kastrup and D. H. Mayer

Institut für Theoretische Physik der Technischen Hochschule Aachen, Aachen, Fed. Rep. Germany

(*Received March 1, 1973*)

The geometric structure of systems which are invariant under the 15-parameter conformal group of the Minkowski space is investigated. In particular, we analyse the action of the group on the compactified Minkowski space M_c^+ and the 5-dimensional manifold $K^5(E, b)$ of events E and measuring rods b. The properties of these manifolds as homogeneous spaces and the structure of their tangent and cotangent spaces are discussed, too. Finally, we investigate the notion of causality in the context of a conformally invariant world: we show that it is possible to introduce a conformally invariant *local* causal structure which is just the same as that of a Poincaré-invariant one in Minkowski space. Globally, this world is highly acausal, because any two points can be connected by timelike curves.

1. Introduction

Recently, an increasing interest is observed in applications of broken conformal and scale (dilatations) transformations to problems in particle physics ([25], [14]), either in terms of Ward identities for Green's functions or as an asymptotic symmetry at very high energies or — equivalently — at short distances in the sense that in this realm the symmetry-breaking rest masses become negligible ([15], [16]).

A particularly interesting formulation of this hypothesis has been given by Wilson in terms of an asymptotic expansion ([31], [34]) for the product of two fields $A(x)$ and $B(y)$ for $||x-y||\to 0$, where $||x||$ denotes the Euclidean norm. In this expansion the terms with the leading (i.e. strongest) singularities are supposed to have exact scale invariance. Several authors ([1], [7], [8], [21]) have discussed the extension of Wilson's expansion to the whole light cone. However, on the light cone not only the dilatations but, in particular, the more general "special conformal" transformations may be interesting.

Another very promising field for applications of the full conformal group is the analysis of conformally invariant Green's functions ([22]–[24]). Especially, Mack and Todorov proved the absence of ultraviolet singularities for conformally invariant Green's function.

* Work supported by the Bundesministerium für Bildung und Wissenschaft.

However, they used only local properties of the conformal transformation, because, globally, the special conformal transformations are not well-defined: this group may be generated by translations

$$T_c:\ x^j \mapsto x^j + c^j, \quad j=0,1,2,3,$$

and the inversion

$$I_r:\ x^j \mapsto -x^j/x\cdot x, \quad x\cdot x = (x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2.$$

The index r stands for "reciprocal radii". The transformations I_r are singular, because the "light-cone" $x\cdot x=0$ has no image. Since the "special conformal" group $SC_4(c)$ – the index "4" denotes the number of independent group parameters – is defined as

$$SC_4(c):\ x^j \mapsto I_r T_c I_r x^j = (x^j - c^j x\cdot x)/\sigma(x;c),$$

$$\sigma(x;c) = 1 - 2c\cdot x + (c\cdot c)(x\cdot x),$$

it is not well-defined, either, namely, for those x and c for which $\sigma(x;c)=0$.

Another very important and controversial ([3], [6], [9], [17], [18], [19]) problem is the following: As $x\cdot x \mapsto x\cdot x/\sigma(x;c)$ under $SC_4(c)$, and since σ is not positive definite, the sign of $x\cdot x$ may change and spacelike x can be mapped onto timelike ones and vice versa.

One cannot deal with the second problem before having taken care of the first one. We therefore proceed in the following way: We first extend the Minkowski space $M^4=\{E\}$ of space-time events in such a way that the action of the full conformal group $C_{15}(a, \Lambda, \rho, c)$, consisting of the full Poincaré group $P_{10}(a, \Lambda)$, the dilatation $D_1(\rho)$: $x_j \mapsto \rho x_j$, $\rho>0$ and the group $SC_4(c)$ becomes continuous. In doing so we keep two main purposes in mind: On one hand we want the physical operations associated with geometrical measurements in the *flat* Minkowski space to hold also for the extended one and, on the other, we want to use the notions and some results of modern global theory of differentiable manifolds ([11], [19], [32]). In order to combine these two purposes we have to translate the physical measuring operations into the mathematical language of mappings. This analysis and the extension of the Minkowski space is performed in Sections 2–5. The reader, not very familiar with the modern notions of differentiable manifolds, is referred to the textbooks by Kobayashi and Nomizu [19], to the one by Wolf [32] and to that by Helgason [11].

The main results of these sections are:

The extension of the Minkowski space M^4 is achieved by considering ([17]) not only the space-time events E, but also a one-dimensional manifold B of measuring rods b at each event E. The manifold B is diffeomorphic to the multiplicative group $A=\boldsymbol{R}^1-\{0\}$ of real numbers $\boldsymbol{R}^1$. Contrary to the Poincaré-group, the group C_{15} acts in a nontrivial way not only on the manifold $\{E\}$ of space-time events but also on the manifold $B=\{b\}$ of the measuring rods. The result of the action of all elements of C_{15} on $\{E\}$ and $\{b\}$ then gives a manifold $K^5(E, b)$ homeomorphic to $S^1\times S^3\times \boldsymbol{R}^1$ (S^n: n-dimensional unit sphere). The required extension M_c^4 of M^4 is then obtained by the factorization $M_c^4=K^5$ $(E, b)/A$

and is homeomorphic to the parallelizable ([12], p. 183) compact manifold $S^1 \times S^3/Z_2(6)$; $Z_2(6) = \{1_6, -1_6\}$; 1_6 is the unity transformation in $\boldsymbol{R}^6$.

In Section 6 the connection of the manifolds $K^5(E, b)$ and M_c^4 with homogeneous spaces and some aspects of the topological structure of the conformal group are discussed.

We then turn to the analysis of the main problem of our paper, namely, the notion of "geometrical" causality in the following sense: Let us briefly recall the situation as to the geometrical causality concepts in the framework of the Poincaré group $P_{10}^{\uparrow}(a, \Lambda)$ acting on M^4 with the metric form $g(x, y) = (y-x)\cdot(y-x)$, $x = x(E_1)$, $y = x(E_2)$. Here the "geometrical" part of the causality concept is as follows:

There exist

(i) a $P_{10}^{\uparrow}(a, \Lambda)$-invariant separation of timelike, spacelike and lightlike events and

(ii) a $P_{10}^{\uparrow}(a, \Lambda)$-invariant time-ordering for timelike and lightlike events, the ordering being given by the function $\operatorname{sgn}(x^0(E_2) - x^0(E_1))$. $P_{10}^{\uparrow}(a, \Lambda)$ is the orthochronous Poincaré group. Define $E_2 > E_1$ ("E_2 is later than E_1") if $g(x, y) > 0$ and $x^0(E_2) > x^0(E_1)$. The following important property was proved [33] by Zeeman: Let f be a bijective mapping $M^4 \to M^4$ and call f a *causal automorphism* if it has the property $f(E_2) > f(E_1)$ iff $E_2 > E_1$ for all $E \in M^4$. The set of all causal automorphisms forms the "causality group" $G_{cs}(M^4)$ of the Minkowski space M^4. Zeeman's theorem then asserts

$$G_{cs}(M^4) = (L_6^{\uparrow}(\Lambda) \otimes D_1(\rho)) \otimes T_4(a),$$

where $L_6^{\uparrow}(\Lambda)$ denotes the orthochronous Lorentz group. As the group $SC_4(c)$ is *not* bijective on M^4, we have to go beyond it, namely, to M_c^4.

It turns out that the properties of the tangent spaces $T_q(K^5)$ and $T_{[q]}(M_c^4)$ are very important in the context of our problem of a conformally invariant notion of causality for the following reason (Section 7): The actions of the groups $O(2, 4)$ and $C_{15} = O(2, 4)/Z_2(6)$ on the tangent bundles $T(K^5)$ and $T(M_c^4)$ are such that one can introduce the concept of a conformally invariant "causal structure". This structure is not very interesting globally, because any two points of the manifold can be connected by a timelike signal, and therefore, globally, this world is highly acausal!

2. Structure of the set of events and measuring rods

The purpose of this and the next three sections is twofold: First, to give a rather precise mathematical formulation of the operations associated with measurements by rods in the Minkowski space and to show how the scale and conformal transformations are to be understood as geometrical "gauge" transformations in the framework of a flat space. Second, to discuss the structure of the differentiable manifold $K^5(E, b)$ of events E and measuring rods b and its projection π: $K^5(E, b) \to M_c^4$ onto the compactified Minkowski space M_c^4 and to analyse the differentiable action of the 15-parameter conformal group on these manifolds.

The physical systems we are going to consider ([17]) contain — among other objects

— "events" E in Minkowski space M^4 and "measuring rods" or "bars" b, an event E being given, for instance, by the location in space and time of the decay of a pion into a myon and an antineutrino, whereas a measuring rod is, for instance, given by the "urmeter" in Paris or the Compton wave length of a certain electron. We shall discuss the space of events first:

2.1. Neglecting the "distortions" of space and time by heavy masses, the structure of the space $M^4=\{E\}$ is assumed to be a 4-dimensional flat pseudo-Riemannian manifold, that is to say, there is a homeomorphism φ between the space M^4 and the 4-dimensional pseudo-Euclidean space $\boldsymbol{R}_1^4=\{x=(x^0, \ldots, x^3)\}$ with the metric form $g(x,y)=x^0y^0-\vec{x}\cdot\vec{x}$. The mapping φ: $E\mapsto x\in \boldsymbol{R}_1^4$ can be implemented by exploiting the fact that the tangent space $T_x(\boldsymbol{R}_1^4)$ at $x=\varphi(E)$ can be identified with R_1^4 itself:

$$\sigma_x:\ T_x(\boldsymbol{R}_1^4)=\{t_x=a^j(x)\,\partial_j;\, a^j\in \boldsymbol{R}^1\}$$
$$\to \boldsymbol{R}_1^4=\{(x^0+a^0(x), \ldots, x^3+a^3(x)), x \text{ fixed}\}. \tag{1}$$

The map σ_x is a diffeomorphism and depends on the (arbitrary) point $x\in \boldsymbol{R}_1^4$. If $x=0$, we simply write $\sigma_{x=0}\equiv\sigma$. In relation (1) the notation $\partial_j\equiv\partial/\partial x^j$ and the summation convention have been used. The metric on $T_x(\boldsymbol{R}_1^4)$ is given by the bilinear form

$$\tilde{g}_x=dx^0\otimes dx^0-dx^1\otimes dx^1-dx^2\otimes dx^2-dx^3\otimes dx^3=g_{jk}\,dx^j\otimes dx^k, \tag{2}$$

where the differentials dx^j form a basis of the dual space $T_x^*(\boldsymbol{R}_1^4)$. In this paper the quanti ties g_{jk} will *always* have the values $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$, all other g_{jk} vanish, for all x. With these definitions we have

$$\tilde{g}_x(a^j\partial_j,\, b^j\partial_j)=a^jb_j,$$

because $dx^j(\partial_k)=\delta^j_k$.

The bijective identification map σ_x is typical for a flat space. It can be exploited to implement the map φ through measurements by using the correspondence between the elements of $T_x(\boldsymbol{R}_1^4)$ and the measuring rods which are available at $\varphi^{-1}(x)=E$. This correspondence is very important for our purpose and we shall therefore discuss some properties of measuring rods next.

2.2. The measuring rods available in a given system form a set $B=\{b\}$ which we assume to be nonempty. Here the rods are specified independently of their direction in space-time, i.e. the set B, the structure of which will be discussed later, is at most a 1-dimensional space. On the other hand, if we look upon a measuring rod as a "vector" in space-time, we write $\boldsymbol{b}$. It is clear that it suffices to talk about rods, because any clock is equivalent to a rod by giving the distance a light ray will travel during a certain time interval.

Generally, the rods b or the vector rods $\boldsymbol{b}$ are attached to a certain space-time event E and we write $b(E)=b_E$ or $\boldsymbol{b}(E)=\boldsymbol{b}_E$.

Our basic assumption is that the rods $\boldsymbol{b}_E$ correspond, via a mapping φ_*, to tangent vectors $t_x=t_{\varphi(E)}=\varphi_*(\boldsymbol{b}_E)$ at $x=\varphi(E)\in \boldsymbol{R}_1^4$. Before discussing the implementation of the maps φ and φ_* by physical operations, let us add the following remarks. Suppose a mapping φ_* exists; then denoting by B_E^4 the set $\{b_E\}$, we can define a metric on B_E^4 by

$$l^2(b_E)=g_E(b,b)=\tilde{g}_{x(E)}[\varphi_*(b_E),\varphi_*(b_E)], \tag{3}$$

where $l(b_E)$ is the length of b_E.

As the Minkowski space is flat, we can move a certain vector rod b from one event E_1 to another one E_2 by translating it, space-time rotating it etc., without changing its length. If we denote any such motion (all of them together form the full Poincaré group) by τ, we have

$$\begin{aligned}\tau:\ & b_1(E_1)\mapsto b_2(E_2),\\ & b_1(E_1)\mapsto b_1(E_2),\\ & l[\tau(b_1(E_1))]=l[b_1(E_1)].\end{aligned} \tag{4}$$

Intuitively this means: if one takes a copy $b_1(E_1)$ of the "urmeter" in Paris to a point E_2 in another town, then one has there a vector rod $b_2(E_2)$, which in general will no longer be parallel to $b_1(E_1)$, but which will have the same length as $b_1(E_1)$.

Contrary to the mathematical procedure in the context of differentiable manifolds (see [11], p. 24), where one starts with the mapping φ and then derives the induced mapping φ_*, we shall start with a description of the physical operation implementing φ_* and then define φ in such a way that the usual mathematical relations between φ and φ_* hold. The reason is that coordinate numbers in flat space are determined by employing measuring rods

The map φ_* can be implemented in the following way: Take $x=0=(0,0,0,0)$ in equation (1) and choose an arbitrary event $\mathring{E}$ to be the corresponding event $E(x=0)=\varphi^{-1}(x=0)$ in M^4. Select a rod $\hat{b}\in B$, take it – by a motion τ – to $\mathring{E}$, place one of its endpoints on $\mathring{E}$, and the rod itself by turns along the four independent orthogonal directions – it is assumed that one knows how to do this technically – such that the other endpoint coincides with the events E_j, $j=0,1,2,3$. This procedure gives us four vector rods $\hat{b}_j=\overrightarrow{(\mathring{E},E_j)}$. The mapping φ_* can now be defined by the property

$$\varphi_*(\hat{b}_j)=\partial_j|_{x=0},\quad j=0,1,2,3, \tag{5}$$

and by the requirement that it is linear. This means that $g_E(\hat{b}_j,\hat{b}_k)=g_{jk}$. The rod $\hat{b}$ is called the *unit of length*. In the following we shall denote by $[\mathring{E},\hat{b}]$ a *frame* which consists of an origin $\mathring{E}$ and a choice of orthogonal unit vector rods $\hat{b}_j$ attached to $\mathring{E}$.

The mapping φ_* is one-to-one (injective), but in general not onto (surjective), because, generally, atomic systems do not contain a continuum of rods b with a continuum of Poincaré invariant lengths – Compton wave lengths, for instance. This is an extremely important point for the application of our discussion to realistic physical systems, which in general are not scale invariant. Thus the set B_E^4 is in general not a vector space isomorphic to $T_{x(E)}(R_1^4)$, but only a set which is bijective to a subset $\varphi_*(B_E^4)\subset T_{x(E)}(R_1^4)$. It is, however, possible to *imagine*, by interpolation and extrapolation or, mathematically speaking, by forming the linear hull of the physically realizable set $B^4_{\mathring{E}}$, a vector space $\check{B}^4_{\mathring{E}} \supset B^4_{\mathring{E}}$ of vector rods $\check{b}_E$ which is isomorphic to $T_{x(E)}(R^4_1)$. The extension $\check{\varphi}_*$: $\check{B}^4_{\mathring{E}} \to T_{x(E)}(R^4_1)$ is a vector space isomorphism.

Next we consider the map φ. It can be implemented by using rods b and their idealizations $\check{b} = \xi\hat{b}$, where ξ is some real number. Let E be any event in M^4 and $\check{b}(\mathring{E}, E)$ the vector rod $\check{b}$ with its initial point in $\mathring{E}$ and its endpoint in E. Project $\check{b}$ orthogonally onto the four basic directions as discussed above and denote the four "oriented" lengths $l(\check{b}_j) = \pm|l(\check{b}_j)|$ – i.e. take into account whether the projections lie on the positive or negative part of the four axis – of these four projections $\check{b}_j$ in terms of the unit vector rods $\hat{b}_j(\mathring{E})$ by ξ^j. Then we have a homeomorphism $\varepsilon : E \mapsto \check{b}_{\mathring{E}}$, which depends on $[\mathring{E}, \hat{b}]$:

$$\varepsilon_{[\mathring{E}, \hat{b}]}(E) = \xi^j \hat{b}_{j, \mathring{E}}, \quad \xi^j = l(\check{b}_j). \tag{6}$$

Further, denote by $e_0, \ldots, e_3$, the points $(1, 0, 0, 0), \ldots, (0, 0, 0, 1)$ of R^4_1 and by ω the linear mapping with the property

$$\omega(\hat{b}_j) = e_j. \tag{7}$$

The map ω depends on the choice of $\mathring{E}$ and $\hat{b}_j$, too. We have

$$\omega_{[\mathring{E}, \hat{b}]}(\xi^j \hat{b}_j) = \xi^j e_j, \tag{8}$$

and

$$x^j(E) = \xi^j(E; [\mathring{E}, \hat{b}]). \tag{9}$$

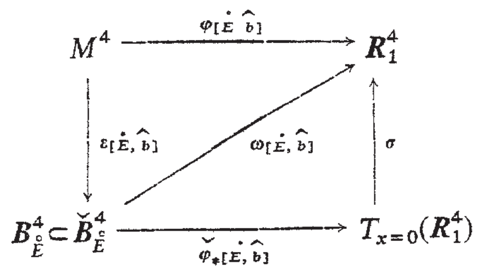
The map φ is therefore given by the composition

$$\varphi_{[\mathring{E}, \hat{b}]} = \omega_{[\mathring{E}, \hat{b}]} \circ \varepsilon_{[\mathring{E}, \hat{b}]}. \tag{10}$$

Since $\sigma(\hat{c}_j) = e_j$, we further have

$$\omega_{[\mathring{E}, \hat{b}]} = \sigma \circ \check{\varphi}_{*[\mathring{E}, \hat{b}]}. \tag{11}$$

The above discussions can be summarized in the following diagram:



All the mappings φ, ε, ω, $\check{\varphi}_*$ and σ are homeomorphisms.

The way we have defined our mappings ε and ω is characteristic for a *flat* space, for in a space with heavy masses, influencing the metric structure, the above identification of lengths of rods on one hand, and the coordinate numbers of events on the other, is no longer possible.

The importance of differentiating between the different mappings or, in other words, the different rules for relating physical operations to the real numbers, will become more evident below. Up to now it may look like making life more complicated.

In case there are other rods $b \neq \hat{b}$ in B, we assume that all such rods are "gauged" in "units of $\hat{b}$" such that

$$b = [1/\kappa_{\hat{b}}(b)]\hat{b}, \quad \kappa \neq 0, \tag{12}$$

where κ is a real number depending on $\hat{b}$ and b. For instance, the Compton wave lengths of all hadrons can be measured in terms of the Compton wave length of the pion. Instead of using the rod $\hat{b}$, we may, of course, implement the mapping φ by means of the rod b (e.g., replace the pion by the nucleon in the above example). In that case we have

$$\varepsilon_{[\mathring{E}, b]}(E) = \eta^j \boldsymbol{b}_j, \tag{13}$$

where

$$g_{\mathring{E}}(\boldsymbol{b}_j, \boldsymbol{b}_k) = 0, \quad j \neq k; \quad \boldsymbol{b}_j = [1/\kappa_{\hat{b}}(b)]\hat{\boldsymbol{b}}_j. \tag{14}$$

A comparison between equations (6), (9) and (13) gives

$$x^j = \eta^j/\kappa. \tag{15}$$

Presently we leave open the question how many rods does the set B contain, i.e. what values may the number κ take. Later we shall admit any real value for κ. This will be a crucial restriction concerning the structure of the systems which are compatible with this assumption.

3. The action of the Poincaré group $P_{10}[\Lambda, a]$ on M^4 and B

We briefly collect some well-known properties we shall need later on in terms of the language we are using. The action of the full 10-parameter Poincaré group $P_{10}[\Lambda, a]$ on M^4 is a mapping $\tau: E \mapsto E'$ such that

$$x^{j\prime} \equiv x^j(E') = \Lambda^j_k x^k(E) + a^j. \tag{16}$$

The numbers $x^{j\prime}$ are determined by the same basic frame $[\mathring{E}, \hat{b}]$ as before, i.e. in the conventional language the map (16) is an "active" transformation.

The mappings (16) of R_1^4 onto itself, induced by the group of motions τ of the Minkowski space M^4, induce a mapping of the tangent spaces: $T_x(R_1^4) \to T_{x'}(R_1^4)$. Let us briefly recall how such an induced map is defined (see [19], p. 8).

Generally, if μ is a mapping $\mu: M \to N$ of the manifolds M and N, such that $\mu(u \in M) = v \in N$ and if $f(v)$ is a differentiable function in a neighbourhood of $v = \mu(u)$, then we have an induced tangent map $\mu_*: T_u(M) \to T_{v=\mu(u)}(N)$ defined by

$$(\mu_* t_u)_v f(v) = t_u(f \circ \mu)(u), \tag{17}$$

where $(f\circ\mu)(u)=f[\mu(u)]$, i.e. the tangent vectors at the point u are being mapped on tangent vectors at $v=\mu(u)$.

Applied to the group of translations $T_4[a]=\{T_a\}$ and the Lorentz transformations $L_6[\Lambda]$, this gives

$$T_4[a]:\ (\partial_j)_x \to [(T_{aj})_* \partial_j]_{x'} = \partial/\partial x^{j\prime}, \tag{18a}$$

$$L_6[\Lambda]:\ (\partial_j)_x \to (\Lambda_* \partial_j)_{x'} = \Lambda_j^k \partial/\partial x^{k\prime}. \tag{18b}$$

According to the mapping (5), these transformations may be considered to be induced by the mappings

$$T_4[a]:\ \boldsymbol{b}_j(E) \mapsto [(T_{aj})_* \boldsymbol{b}_j](E') = \boldsymbol{b}_j(E'), \tag{19a}$$

$$L_6[\Lambda]:\ \boldsymbol{b}_j(E) \mapsto (\Lambda_* \boldsymbol{b}_j)(E') = \Lambda_j^k \boldsymbol{b}_k(E'). \tag{19b}$$

Because of the relations (4), we have on the other hand

$$P_{10}[\Lambda, a]:\ b(E) \mapsto b(E'), \quad b \mapsto b \in B. \tag{20}$$

In this sense the action of the Poincaré group on B, i.e. on the set of rods independent of the event E, is the identity transformation.

4. Gauge transformations of the set of measuring rods B and the Minkowski space M^4

Because of the property (20), Poincaré invariance does not require more than the existence of just one $b \in B$. There may be more, of course. If there are more than just one, we may consider mappings $B \to B$. To be more specific: Let B_E be the set of all measuring rods of the system which are in principal available at the event E, and $B_{E'}$, the corresponding set at the event E'. We assume $B_E = B_{E'}$, because rods may be moved around freely in our flat space-time, without changing their length. Among the possible mappings (so-called *gauge transformations*) of the type

$$B_E \to B_E, \quad b_E \mapsto b'_E = \beta(E)\, b_E, \tag{21}$$

which induce a map $E \to E'$ in a way to be specified below, we have the following ones:

4.1. *Dilatations*

Let us assume the number $\beta(E)$ to be independent of the event E and to be any positive real number, i.e. $\beta(E) \in \boldsymbol{R}^+ = \{\rho > 0\}$.

This assumption implies that the system under consideratation contains a *continuous* set of measuring rods, in other words, the number κ in (12) may be any positive real number. We denote the corresponding set B by B^+. The set B^+ then has the structure of a 1-dimensional differentiable manifold. The mapping $B^+ \to B^+$:

$$D_1[\rho]:\ b \mapsto b' = \rho b, \quad \rho > 0, \tag{22}$$

is called a *dilatation* or *scale* transformation. It can be a genuine symmetry operation only for those systems which contain a continuous set of Poincaré invariant "units of length", or, in other words, for those systems for which $B^4 = \check{B}^4$. Thus it is in general not a symmetry operation for atomic systems, because, for instance, the set of all Compton wave lengths of atomic particles is not a continuous one.

We want to emphasize that we are discussing only the so-called *active point* of view in this paper, i.e. we shall not discuss "passive" scale transformations like the ones associated with the spontaneous breakdown of scale or conformal invariance ([5], [26]).

The mapping $\hat{b}_{\mathring{E}} \mapsto \hat{b}'_{\mathring{E}} = \rho \hat{b}_{\mathring{E}}$ induces a mapping $\hat{b} \mapsto \rho \hat{b}$ of $B^+_{\mathring{E}}$ into itself. Combining this map with the map ε discussed before, we get the mapping

$$E = \overset{-1}{\varphi}_{[\mathring{E},\hat{b}]}(x) \mapsto E' = \overset{-1}{\varphi}_{[\mathring{E},\hat{b}']}(x) = \overset{-1}{\varphi}_{[\mathring{E},\hat{b}]}(\rho x) \tag{23}$$

of the Minkowski space M^4 into itself. On R^+_1 we have

$$x^j \to \rho x^j. \tag{24}$$

From equations (12), (13) and (22) we obtain

$$\begin{gathered} \varepsilon_{[\mathring{E},b']}(E') = \eta^j b'_j, \\ b' = [1/\kappa_{\hat{b}}(b)]\,\hat{b}' = [\rho/\kappa_{\hat{b}}(b)]\,\hat{b}, \\ \kappa_{\hat{b}}(b') = \rho^{-1}\kappa_{\hat{b}}(b); \quad \kappa_{\hat{b}'}(b') = \kappa_{\hat{b}}(b); \quad \eta^j(E') = \eta^j(E). \end{gathered} \tag{25}$$

Our remarks above indicate that we can define a 5-dimensional differentiable manifold

$$M^4 \times B^-$$

which is homeomorphic to the space

$$N^5_+ = \{\eta = (\eta^0, \ldots, \eta^3, \kappa > 0)\} = R^4_1 \times R^+. \tag{26}$$

The space of events M^4 is then homeomorphic to the space $\{[\eta]\}$ of all equivalence classes $[\eta] = \{\eta, \overset{1}{\eta} \hat{=} \overset{2}{\eta} \between \overset{1}{\eta} = \alpha \overset{2}{\eta}, \alpha > 0\}$. This follows immediately from (15). The projection

$$M^4 \times B^+ \to M^4 \sim N^5_+/A^+, \tag{27}$$

where $A^+ = \{\alpha > 0\}$ is the multiplicative group of positive real numbers, will be denoted by $\hat{\pi}_\kappa$.

The introduction of the manifold $M^4 \times B^-$ may seem somewhat artificial. Its importance will become clearer below in the context of the special conformal transformations where the manifold of events and measuring rods is no longer a natural direct product, but a fibre bundle (see [19], p. 50), i.e. only *locally* a direct product.

Notice that the actio i of the group A^+ o1 N^5_+ or M^4 is differe .t from that of $D_1[\rho]$, although the two groups are isomorphic.

Up to now our discussion has mainly been a translation of well-known physical properties and operations into the modern language of the theory of differentiable manifolds, nothing more. However, the use of this language will turn out to be quite useful for the mathematical and physical results we are aiming at, in the context of the full conformal group, where life becomes more complicated.

4.2. *Special conformal transformations*

Next we allow the factor β_E in (21) to depend in a non-trivial way on the event E.

In general, one might think that the factors β_E in equation (21) are arbitrary functions of E or $x=\varphi(E)$, respectively. However, this would imply that the new metric $g_{ij}\to\bar{g}_{ij}=\beta_E^2 g_{ij}$, generally, will have a non-vanishing curvature ([30]), i.e. the Riemannian space thus obtained is not flat. Then it follows that we can no longer determine coordinate numbers like we did before in connection with the mappings ε and ω. However, there is a subclass of nontrivial functions β_E which do leave the flat character of the metric unchanged and these are the very ones which are induced by the so-called *special conformal transformations* of the Minkowski space. For the proofs of these properties we refer to the literature ([30], [4]). In our language these results may be described as follows:

The special conformal transformations can be generated by the inversion

$$\tilde{I}_r\colon\ b_{E'}\mapsto b'_E=-(x\cdot x)^{-1}b_E,\qquad x\cdot x\neq 0, \tag{28}$$

where $x^j=x^j(E)$.

This map of the rods is associated with the following map of events:

The coordinate numbers $x^j(E')\equiv x^{j'}$ are determined according to the rules ε and ω in the following way: Let us assume that the coordinates $x(E)$ were determined by the map $\varepsilon_{[\mathring{E},\hat{b}]}$ with $\hat{b}_E=\hat{b}_{\mathring{E}}$ for all E, i.e. we start with the same unit of length at each event E. We next move the rod $\hat{b}'_E=-(x\cdot x)^{-1}\hat{b}_E$ to the event $\mathring{E}$. This action does not change its length, because the physical space is supposed to be flat. Then we form the frame $[\mathring{E},\hat{b}'_E]$ by placing four copies $\hat{\boldsymbol{b}}'_j$ along the same four basic axis as we did before with $\hat{\boldsymbol{b}}_j$. The coordinates $x^{j'}$ and the corresponding events E' are then determined by the rule

$$\varepsilon_{[\mathring{E},\hat{b}'_E]}(E')=x^j\hat{\boldsymbol{b}}'_j=-(x\cdot x)^{-1}x^j\hat{\boldsymbol{b}}_j. \tag{29}$$

Thus we have on $\boldsymbol{R}_1^4$ the mapping

$$I_r\colon\ x^j(E)\mapsto x^j(E')\equiv x^{j'}=-x^j(x\cdot x)^{-1}. \tag{30}$$

Furthermore, according to equation (17) we have the following induced map $T_x(\boldsymbol{R}_1^4)\to T_{x'}(\boldsymbol{R}_1^4)$ of tangent spaces:

$$\partial/\partial x^j\mapsto(I_{r*}\partial_j)_{x'}=(x\cdot x)^{-2}(2x_jx^k-\delta_j^k x\cdot x)\,\partial/\partial x^{k'}. \tag{31}$$

The linearity of the mapping ω_x^{-1} then gives

$$\hat{\boldsymbol{b}}_j(E)\mapsto\hat{\boldsymbol{b}}'_j(E')=(x\cdot x)^{-2}(2x_jx^k-\delta_j^k x\cdot x)\,\hat{\boldsymbol{b}}_k(E'), \tag{32}$$

where the vector rod $\hat{b}_k(E')$ at E' may be considered to be the image of the vector rod $\hat{b}_k(E)$ obtained by parallel translation from E to E'.

Comparing the relations (29) and (32) with the mappings σ and ω as defined by equations (1) and (7), we see that the relation

$$I_{r*}\sigma^{-1}(e_j)=\sigma^{-1}(I_r e_j)$$

does not hold any more, the reason being the nonlinearity of the mapping I_r on $\boldsymbol{R}_1^4$.

Usually, one starts with the mapping (30) and then deduces the relation (31) etc. We think that our unconventional way stresses the nature of the physical operations in a more transparent way. The map (32) has the property

$$g_{E'}(\hat{\boldsymbol{b}}'_{E'}, \hat{\boldsymbol{b}}'_{E'})=\beta_E^2 g_E(\hat{\boldsymbol{b}}_E, \hat{\boldsymbol{b}}_E), \qquad \beta_E=-(x\cdot x)^{-1},$$

which expresses the fact that the mapping is a conformal one.

The mapping I_r is not an element of the identity component of the full conformal group of the Minkowski space M^4. However, the transformation $I_r T_c I_r$, where T_c is a translation, does belong to this component. Since the action of a translation on M^4 can be characterized by

$$\varepsilon_{[\mathring{E}', \hat{b}_E]}(E')=x^j\hat{\boldsymbol{b}}_j, \qquad \text{or} \qquad \varepsilon_{[\mathring{E}, \hat{b}_E]}(E')=(x^j+a^j)\,\hat{\boldsymbol{b}}_j, \tag{33}$$

thus, combined with equations (29) and (31), this gives

$$SC_4[c]:\ b_E\mapsto b'_E=[1/\sigma(x;c)]\,b_E \tag{34}$$

and

$$x^j\hat{\boldsymbol{b}}_j=\varepsilon_{[\mathring{E}, \hat{b}]}(E)\mapsto \varepsilon_{[\mathring{E}, \hat{b}'_E]}(E')=(x^j-c^j x\cdot x)\,\hat{\boldsymbol{b}}'_j, \tag{35}$$

or

$$\varepsilon_{[\mathring{E}, \hat{b}]}(E')=[(x^j-c^j x\cdot x)/\sigma(x;c)]\,\hat{\boldsymbol{b}}_j,$$

where

$$\sigma(x;c)=1-2c\cdot x+(c\cdot c)(x\cdot x).$$

These mappings imply the mapping

$$x^j\mapsto(x^j-c^j x\cdot x)/\sigma(x;c) \tag{36}$$

on $\boldsymbol{R}_1^4$.

According to the literature already quoted ([4], p. 230 and 284 ff.) the full Poincaré group $P_{10}[\Lambda, a]$, the dilatations $D_1[\rho]$ and the special conformal transformations $SC_4[c]$ form the full conformal group $C_{15}[\Lambda, a, \rho, c]$ which is the most general transformation group of the Minkowski space M^4 having the property

$$g_{E'}(\boldsymbol{b}'_{E'}, \boldsymbol{b}'_{E'})=\beta_E^2 g_E(\boldsymbol{b}_E, \boldsymbol{b}_E),$$

where $E\mapsto E'$ and $\boldsymbol{b}_E\mapsto\boldsymbol{b}'_{E'}$.

However, this statement is unsatisfactory in the sense that the function β_E may be discontinuous, and therefore, the action of C_{15} on M^4 is not continuous. Our next step will be to take care of this.

5. The manifold $K^5(E, b)$ of events and measuring rods

In equation (28) we had to require $x \cdot x \neq 0$ in order to have a well-defined expression. This shows that the inversion I_r and the special conformal transformations (35) are globally not well-defined in the Minkowski space. In order to get an everywhere continuous action of the 15-parameter conformal group, we have to extend the Minkowski space by "adding points at infinity". In detail this means the following:

Consider the example: Let us assume that the rod $\hat{b}$ in equation (12) is the Compton wave length $\lambda_c(e)$ of an electron. If, for one reason or the other, the rod b is the Compton wave length $\lambda_c(\gamma)$ of a photon, then we have $\kappa_{\lambda_c(e)}[\lambda_c(\gamma)]=0$. Such values of κ are obtained as a result of the mapping $\tilde{I}_r$, as can be seen from (12) and (28):

$$b_E \mapsto b'_E = [1/\kappa_{\hat{b}'}(b')]\,\hat{b}'_E = -[1/\kappa_{\hat{b}}(b)\, x \cdot x)]\,\hat{b}_E . \tag{37}$$

We have

$$\kappa_{\hat{b}}(b'_E) = -\kappa_{\hat{b}}(b_E)\, x \cdot x = -\eta^j \eta_j / \kappa_{\hat{b}}(b_E),$$
$$\eta^j(E') = \eta^j(E), \tag{38}$$

and therefore, $\kappa_{\hat{b}}(b'_E)=0$ for $x \cdot x = 0$. In addition we see that the inversion $\tilde{I}_r$ induces negative values of κ. As these negative values of κ can be dealt with rather easily, let us discuss this case first: a "negative" rod is a rod gauged in, for instance, "$(-\text{cm})$", if the corresponding unit of length is "cm". We denote by B_0 the set of all positive and negative rods b^+ and $b^- = -b^+$, where our previous set B^+ is identical with the set $\{b^+\}$. Notice that the "zero" rod $b^+ \oplus b^-$ is not contained in B_0, because this would mean $\kappa_{\hat{b}}(b^+ \oplus b^-) = \infty$.

The manifold $M^4 \times B_0$ of events and measuring rods is homeomorphic to the open set

$$N^5_{(\kappa)} = \boldsymbol{R}^4_1 \times \boldsymbol{R}^{(\kappa)}_0 = \{(\eta^0, \dots, \eta^3; \kappa \neq 0), (\eta^j) \in \boldsymbol{R}^4_1, \kappa \in \boldsymbol{R}_0\}, \tag{39}$$

where $\boldsymbol{R}_0 = \boldsymbol{R}^1 - \{0\}$. If we want to emphasize that $\boldsymbol{R}_0$ is the multiplicative group of real numbers, we shall write $\boldsymbol{R}_0 = A = \{\alpha \neq 0\}$.

The Minkowski space $M^4 \equiv M^4_{(\kappa)}$ is now homeomorphic to the quotient space $N^5_{(\kappa)}/A$, i.e. the Minkowski space is homeomorphic to a subspace of a projective space to be specified below.

In order to deal with the case $\kappa_{\hat{b}}=0$, let us first denote the elements of B_0 by $b^{(\kappa)}$ and B_0 itself by $B^{(\kappa)}_0$ and let us define, for $x \cdot x \neq 0$, the rods

$$b^{(\lambda)}_E = (x \cdot x)\, b^{(\kappa)}_E . \tag{40}$$

If the coordinate numbers of the event E, as determined according to the rule $\varepsilon_{[\dot{E},\hat{b}_E^{(\lambda)}]}$ by means of the rod $\hat{b}_E^{(\lambda)}=(x\cdot x)\hat{b}_E^{(\kappa)}$, are the numbers y^j, then we have

$$y^j=x^j/x\cdot x;\qquad x^j=y^j/y\cdot y,\qquad j=0,1,2,3, \tag{41}$$

where $x\cdot x\neq 0$, $y\cdot y\neq 0$. Instead of equation (12) we can now write

$$b=[1/\kappa_{\hat{b}}(b)]\,\hat{b}^{(\kappa)}=[1/(\kappa_{\hat{b}}(b)\,x\cdot x)]\,\hat{b}^{(\lambda)}=[\kappa_{\hat{b}}(b)/\eta^j\eta_j]\,\hat{b}^{(\lambda)}, \tag{42}$$

where the index "$\hat{b}$" of κ always means "$\hat{b}^{(\kappa)}$".

If we define on $N^5_{(\kappa)}$

$$\lambda_{\hat{b}}=x\cdot x\kappa_{\hat{b}}=\eta^j\eta_j/\kappa_{\hat{b}}, \tag{43}$$

we have, for $\lambda_{\hat{b}}\neq 0$,

$$b=(1/\lambda_{\hat{b}}(b))\,\hat{b}^{(\lambda)}\qquad\text{and}\qquad y^j=\eta^j/\lambda_{\hat{b}}.$$

The idea of extending the Minkowski space $M^4_{(\kappa)}$ by adding events "at infinity" may be explained as follows. Using the equations (41), we describe all events with $x\cdot x\neq 0$ by the coordinates y. Then we allow the coordinates y to run through all elements of the space $\boldsymbol{R}^4_1=\{y,\,y\cdot y=y^jy_j\}$ in the same way the coordinates x run through $\boldsymbol{R}^4_1=\{x\}$ as described before. By employing the rods $\hat{b}_E^{(\lambda)}$ and the rules ε, ω etc. in the same way as in the case of the space $M^4_{(\kappa)}$, we obtain a manifold of events $M^4_{(\lambda)}$ and a manifold $B_0^{(\lambda)}$ of measuring rods which, according to equations (43) and (41), have common elements with $M^4_{(\kappa)}$ and $B_0^{(\kappa)}$, namely the intersections $M^4_{(\kappa)}\cap M^4_{(\lambda)}$ and $B_0^{(\kappa)}\cap B_0^{(\lambda)}$, but which in addition contain events and rods which are not in $M^4_{(\kappa)}$ and $B_0^{(\kappa)}$. namely those for which $y\cdot y=0$.

The inversion I_r *alone*, i.e. disregarding all other conformal transformations, is now well defined on $M^4_{(\kappa)}\cup M^4_{(\lambda)}$ in terms of the mappings

$$E\mapsto E';\quad x^j(E')=-y^j(E),\quad E\in M^4_{(\lambda)};\quad y^j(E')=-x^j(E),\quad E\in M^4_{(\kappa)}.$$

In terms of the coordinates η^j, κ and λ the introduction of the manifolds $M^4_{(\lambda)}$ and $B_0^{(\lambda)}$ means the following:

Let us keep $\hat{b}^{(\lambda)}$ fixed for a moment. For $\kappa_{\hat{b}}\to 0$, the number $\lambda_{\hat{b}}$ can stay finite $\neq 0$, if $\eta^j\eta_j\to 0$ accordingly. We see that we can define a manifold $M^4_{(\lambda)}\times B_0^{(\lambda)}$ homeomorphic to the space

$$N^5_{(\lambda)}=\boldsymbol{R}^4_1\times\boldsymbol{R}_0=\{(\eta^0,\dots,\eta^3;\ \lambda\neq 0),\ (\eta^j)\in\boldsymbol{R}^4_1,\ \lambda\in\boldsymbol{R}_0\}.$$

We define $\kappa=\eta^j\eta_j/\lambda$ on $N^5_{(\lambda)}$; then we have for the coordinates η^j, κ, λ on $(M^4_{(\kappa)}\times B_0^{(\kappa)})\cup$ $\cup\,(M^4_{(\lambda)}\times B_0^{(\lambda)})$ the relation

$$\lambda(b_E)\cdot\kappa(b_E)=\eta^j(E)\,\eta_j(E). \tag{44}$$

According to the equations (37) and (40), the inversion I_r can now be characterized on

$(M^4_{(\kappa)} \times B_0^{(\kappa)}) \cup (M^4_{(\lambda)} \times B_0^{(\lambda)})$ as

$$\begin{gathered} \hat{b}^{(\kappa)} \mapsto \hat{b}^{(\kappa)}, \quad \hat{b}^{(\lambda)} \mapsto \hat{b}^{(\lambda)}, \\ \eta^j(E) \mapsto \eta^j(E') = \eta^j(E), \\ \kappa_{\hat{b}} \mapsto -\lambda_{\hat{b}}, \quad \lambda_{\hat{b}} \mapsto -\kappa_{\hat{b}}. \end{gathered} \tag{45}$$

However, whereas the inversion I_r is now well defined on $M^4_{(\kappa)} \cup M^4_{(\lambda)}$, the translations are not. For they leave $M^4_{(\kappa)}$ invariant. But those elements of $M^4_{(\lambda)}$ which correspond to events $\kappa=0$ or $y\cdot y=0$ may be mapped outside $M^4_{(\lambda)}$, and therefore, outside $M^4_{(\kappa)} \cup M^4_{(\lambda)}$. This can be seen immediately from the action

$$x^j \mapsto x^j + a^j, \qquad y^j \mapsto (y^j + a^j y\cdot y)/(1 + 2a\cdot y + a\cdot a y\cdot y)$$

of the translations on the coordinates x and y.

In order to discuss this situation we proceed as follows: First we define for the coordinates of $(M^4_{(\kappa)} \times B_0^{(\kappa)}) \cup (M^4_{(\lambda)} \times B_0^{(\lambda)})$

$$\eta^4 = \tfrac{1}{2}(\kappa+\lambda), \quad \eta^5 = \tfrac{1}{2}(\lambda-\kappa); \quad \tilde{\kappa} = \eta^1 - \eta^0, \quad \tilde{\lambda} = \eta^1 + \eta^0, \tag{46}$$

and for $(x^1 - x^0) \neq 0$, or $(x^1 + x^0) \neq 0$ we introduce the rods

$$\hat{b}_E^{(\tilde{\kappa})} = (x^1 - x^0)\,\hat{b}_E^{(\kappa)}, \quad \hat{b}_E^{(\tilde{\lambda})} = (x^1 + x^0)\,\hat{b}_E^{(\kappa)}. \tag{47}$$

When determining the coordinates y, we selected for each event E a certain rod $\hat{b}_E^{(\lambda)}$ depending on the coordinates $x(E)$ but kept the origin $\mathring{E}$ of the frame $[\mathring{E}, \hat{b}_E^{(\lambda)}]$ fixed. Now we introduce for each event $E \in M^4_{(\kappa)}$ a new origin $\mathring{E}_x$ with the coordinates

$$\mathring{x}(\mathring{E}_x) = (x^0 - \tfrac{1}{2}(x\cdot x - 1), x^1 - \tfrac{1}{2}(x\cdot x + 1), 0, 0)$$

and determine the coordinates $\tilde{x}^j$ according to the rules ε, etc., with the frames $[\mathring{E}_x, \hat{b}_E^{(\tilde{\kappa})}]$. In this way we get for $(x^1 - x^0) \neq 0$:

$$\begin{gathered} \tilde{x}^0 = \frac{1}{2}\frac{x\cdot x - 1}{x^1 - x^0}, \quad \tilde{x}^1 = \frac{1}{2}\frac{x\cdot x + 1}{x^1 - x^0}, \quad \tilde{x}^2 = \frac{x^2}{x^1 - x^0}, \quad \tilde{x}^3 = \frac{x^3}{x^1 - x^0}, \\ x^0 = \frac{1}{2}\frac{\tilde{x}\cdot\tilde{x} - 1}{\tilde{x}^1 - \tilde{x}^0}, \quad x^1 = \frac{1}{2}\frac{\tilde{x}\cdot\tilde{x} + 1}{\tilde{x}^1 - \tilde{x}^0}, \quad x^2 = \frac{\tilde{x}^2}{\tilde{x}^1 - \tilde{x}^0}, \quad x^3 = \frac{\tilde{x}^3}{\tilde{x}^1 - \tilde{x}^2}. \end{gathered} \tag{48}$$

We now extend the manifold $M^4_{(\kappa)} \cup M^4_{(\lambda)}$ by letting the coordinates $\tilde{x}^j$ run through the space $R^4_1 = \{\tilde{x}, \tilde{x}\cdot\tilde{x} = \tilde{x}^j\tilde{x}_j\}$ as we did before with the coordinates y. This gives us a manifold $M^4_{(\tilde{\kappa})}$ which has a nonempty intersection with $M^4_{(\kappa)} \cup M^4_{(\lambda)}$ and the union $M^4_{(\kappa)} \cup M^4_{(\lambda)} \cup$ $\cup\, M^4_{(\tilde{\kappa})}$ is a genuine extension of $M^4_{(\kappa)} \cup M^4_{(\lambda)}$.

However, this extension is still not large enough for our purpose. For there are homogeneous Lorentz transformations Λ such that $(\Lambda x)^1 = (\Lambda x)^0$. Such transformations map an element of $M^4_{(\tilde{\kappa})}$ which is not in $M^4_{(\kappa)} \cup M^4_{(\lambda)}$ outside $M^4_{(\tilde{\kappa})}$ but not into $M^4_{(\kappa)} \cup M^4_{(\lambda)}$, because $M^4_{(\kappa)}$ and $M^4_{(\lambda)}$ are invariant under homogeneous Lorentz transformations.

For these reasons we define coordinates $\tilde{y}$ by means of the rods $\hat{b}_E^{(\tilde{\lambda})}$ and by using the frames $[\mathring{E}_x, \hat{b}_E^{(\tilde{\lambda})}]$, where $\mathring{E}_x$ is the same as above. In this way we get a fourth manifold $M^4_{(\tilde{\lambda})}$ which again contains events not contained in the previous ones. With $z=(\kappa, \lambda, \tilde{\kappa}, \tilde{\lambda})$, we denote by M_c^4 the union

$$M_c^4 = \bigcup_{\alpha \in Z} M^4_{(\alpha)} .$$

It can now be shown that all elements of the full conformal group $C_{15}[\Lambda, a, \rho, c]$ act continuously on M_c^4. In order to see this we first give the connection between the local coordinates $\tilde{x}$ and $\tilde{y}$ and the coordinates η^μ, $\mu=0, \ldots, 5$. It is easy to see that

$$\begin{aligned} &\tilde{x}^0=\eta^5/\tilde{\kappa}; \quad \tilde{x}^1=\eta^4/\tilde{\kappa}, \quad \tilde{x}^2=\eta^2/\tilde{\kappa}, \quad \tilde{x}^3=\eta^3/\tilde{\kappa}, \quad \tilde{\kappa}\neq 0, \\ &\tilde{y}^0=\eta^5/\tilde{\lambda}, \quad \tilde{y}^1=\eta^4/\tilde{\lambda}, \quad \tilde{y}^2=\eta^2/\tilde{\lambda}, \quad \tilde{y}^3=\eta^3/\tilde{\lambda}, \quad \tilde{\lambda}\neq 0, \end{aligned} \tag{49}$$

where the coordinates η^5, η^4, η^2, η^3 have been determined by the rods

$$b=[1/\tilde{\kappa}\mathfrak{z}(b)]\hat{b}_E^{(\tilde{\kappa})}, \qquad b=[1/\tilde{\lambda}\mathfrak{z}(b)]\hat{b}_E^{(\tilde{\lambda})},$$

respectively, and by using the origin $\mathring{E}_x$ as defined above.

As before we can define the product manifolds

$$M^4_{(\tilde{\kappa})} \times B_0^{(\tilde{\kappa})}, \qquad M^4_{(\tilde{\lambda})} \times B_0^{(\tilde{\lambda})}$$

which are homeomorphic to the sets

$$\begin{aligned} N^5_{(\tilde{\kappa})} &= \{(\eta^2, \eta^3, \eta^4, \eta^5)\in \boldsymbol{R}_1^4, \tilde{\kappa}\in \boldsymbol{R}_0\}, \\ N^5_{(\tilde{\lambda})} &= \{(\eta^2, \eta^3, \eta^4, \eta^5)\in \boldsymbol{R}_1^4, \tilde{\lambda}\in \boldsymbol{R}_0\}. \end{aligned}$$

On $N^5_{(\tilde{\kappa})}$ we define

$$\tilde{\lambda}=(1/\tilde{\kappa})[(\eta^5)^2-(\eta^2)^2-(\eta^3)^2-(\eta^4)^2]$$

and $\tilde{\kappa}$ on $N^5_{(\tilde{\lambda})}$ accordingly. Then we have on each $M^4_{(\alpha)} \times B_0^{(\alpha)}$, $\alpha \in Z$, the relation (44).

According to our construction, at least one of the four numbers κ, λ, $\tilde{\kappa}$, $\tilde{\lambda}$ is nonzero. This means $(\eta^0, \ldots, \eta^5)\neq 0$ and even $(\eta^0, \eta^5)\neq 0$, $(\eta^1, \ldots, \eta^4)\neq 0$. On the other hand, each value of the homogeneous coordinates η^μ, $\mu=0, \ldots, 5$, $(\eta)\neq 0$, compatible with the relation (44) indeed occurs because of the values the coordinates x, y, $\tilde{x}$ and $\tilde{y}$ can assume. However, the action of $C_{15}[\Lambda, a, \rho, c]$ on the coordinates η is such that the transformed coordinates η' are linear combinations of the original ones, with continuous (with respect to the group parameters) coefficients (see above).

Furthermore, the quadratic form $\eta^j\eta_j-\kappa\lambda$ is even invariant under this action. From our construction, then, it follows that the manifold M_c^4 is invariant, too, under the full group $C_{15}[A, a, \rho, c]$ and that its action is continuous.

The sets $N_{(\alpha)}^5$, $\alpha \in z$, are local coordinate systems of the differentiable manifold

$$K^5(E, b) = \bigcup_{\alpha \in z} M_{(\alpha)}^4 \times B_0^{(\alpha)} \tag{50}$$

of events and measuring rods. Because of equation (44) this manifold is homeomorphic to the submanifold

$$Q_0^5 = \{q = (\eta^0, \dots, \eta^5), (\eta^0)^2 - (\eta^1)^2 - \dots - (\eta^4)^2 + (\eta^5)^2 = 0\}$$

of the 6-dimensional space $\boldsymbol{R}_0^6$ with the quadratic form

$$g_{\mu\nu}\eta^\mu\eta^\nu, \qquad g_{00} = g_{55} = -g_{11} = - \dots - g_{44} = 1; \qquad g_{\mu\nu} = 0, \qquad \mu \neq \nu,$$

and the origin $q = (0, \dots, 0)$ deleted.

We have the relation

$$M_c^4 = K^5(E, b)/A. \tag{51}$$

Our discussion shows that *the manifold $K^5(E, b)$ is the bundle space of a principal fiber bundle $K^5(M_c^4, A)$ with base space M_c^4 and structure group A*. This means that *globally* we have a nontrivial generalization of the space of events and measuring rods compared to the direct product in Minkowski space.

The global topological structures of the manifolds K^5 and M_c^4 can be characterized as follows: If we write the condition for the points q of the space Q_0^5 as

$$(\eta^0)^2 + (\eta^5)^2 = (\eta^1)^2 + \dots + (\eta^4)^2,$$

we immediately see that

$$Q_0^5 \sim S^1 \times S^3 \times \boldsymbol{R}^+, \tag{52}$$

where S^n denotes the n-dimensional unit sphere. As $\boldsymbol{R}^+$ and S^3 are simply connected, whereas S^1 is connected with the infinite cyclic fundamental group, Q_0^5 is infinitely connected.

As for M_c^4, let $Z_2(6)$ be the group $\{\pm 1_6\}$, where 1_6 is the unity transformation in $\boldsymbol{R}_0^6$. We recall that the 5-dimensional projective space $\boldsymbol{P}^5$ may be represented in the following two ways:

$$\boldsymbol{P}^5 \sim \boldsymbol{R}_0^6/A \sim S^5/Z_2(6).$$

Similarly we have

$$M_c^4 \sim Q_0^5/A \sim (S^1 \times S^3)/Z_2(6). \tag{53}$$

This can be seen immediately by taking the representative

$$(\eta^0)^2 + (\eta^5)^2 = 1 = (\eta^1)^2 + \dots + (\eta^4)^2$$

of the equivalence class $[q] = \{\overset{1}{q} \triangleq \overset{2}{q} \between \overset{1}{q} = \alpha \overset{2}{q}, \ \alpha \in A\}$.

From (53) we see that M_c^4 is compact and that the torus $S^1 \times S^3$ is a twofold covering of M_c^4. The compactified Minkowski space M_c^4 is therefore also infinitely connected and has the universal covering space $R \times S^3$.

Without referring to the 6-dimensional embedding, we can rewrite the relation (53) in the following way: Denote by $\mathrm{Dg}[Z_2(2) \times Z_2(4)]$ the subgroup of $Z_2(2) \times Z_2(4)$ with the elements $1_2 \times 1_4$, $(-1_2) \times (-1_4)$. Then we have the isomorphism

$$Z_2(6) \simeq \mathrm{Dg}[Z_2(2) \times Z_2(4)],$$

and we can rewrite the relation (53) as

$$M_c^4 \sim (S^1 \times S^3)/\mathrm{Dg}[Z_2(2) \times Z_2(4)].$$

One further remark: If we consider the local coordinate systems $N_{(\alpha)}^5$, $\alpha \in z$, as charts of the manifold Q_0^5, we write $\hat{N}_{(\alpha)}^5$ and have

$$Q_0^5 = \bigcup_{\alpha \in z} \hat{N}_{(\alpha)}^5. \tag{54}$$

The discussion of this section is somewhat related to paper [20] by Kuiper.

6. The manifolds $K^5(E, b)$ and M_c^4 as homogeneous spaces

It is easy to verify that the action of the transformations $P_{10}[A, a]$, $D_1[\rho]$, I_r and $SC_4[c]$ on the coordinates η^μ leaves not only the quadric $\eta^\mu \eta_\mu = 0$ but the form $\eta^\mu \eta_\mu$ itself invariant. On the other hand, we shall prove in detail in the next section that relations like

$$dx^j \otimes dx_j = \frac{1}{\kappa^2} d\eta^\mu \otimes d\eta_\mu, \tag{55}$$

hold, where $dx^j \in T_x^*(R_1^4(\kappa))$ and $(d\eta^\mu) \in T_q^*(Q_0^5)$, $R_1^4(\alpha) = \varphi(M_{(\alpha)}^4)$. Equation (55) shows that any linear transformation which leaves the quadratic forms $d\eta^\mu \otimes d\eta_\mu$ and $\eta^\mu \eta_\mu$ invariant induces a conformal transformation on $R_1^4(\kappa)$. The same holds for the other spaces $R_1^4(\alpha)$. Thus any element $w \in O(2, 4)$ induces a conformal transformation on M_c^4. However, the action of $O(2, 4)$ on M_c^4 is only almost effective, because the subgroup $Z_2(6) \subset O(2, 4)$ acts as the identity transformation on M_c^4. As $Z_2(6) = O(2, 4) \cap A(6)$, where $A(6)$ is the multiplicative group of real numbers $\neq 0$ acting on R_0^6, the group $Z_2(6)$ is the kernel of the action in question. Thus the conformal group $C_{15}[A, a; \rho, c]$ of the space M_c^4 can be identified with the factor group $O(2, 4)/Z_2(6)$.

Let us recall very briefly some properties of the group $O(2, 4)$. If $\xi \equiv (\xi^0, \ldots, \xi^5) \in R_2^6$ and $w = (w_\nu^\mu) \in O(2, 4)$, then the group $O(2, 4)$ acts on R_2^6 in such a way that

$$\xi^\mu \mapsto \hat{\xi}^\mu = w_\nu^\mu \xi^\nu, \quad \hat{\xi}^\mu \hat{\xi}_\mu = \xi^\mu \xi_\mu, \quad \xi_\mu = g_{\mu\nu} \xi^\nu, \tag{56}$$

where $g_{00}=g_{55}=-g_{11}=\ldots=-g_{44}=1$, all other $g_{\mu\nu}$ vanish. The equations (56) imply

$$w^T g w = g ,$$

where w^T is the transposed matrix of w.

The group $O(2, 4)$ consists ([2]) of four components which are defined as follows:

$$SO^{\uparrow}(2, 4): \det(w)=+1, \quad \operatorname{sign}(w_0^0 w_5^5 - w_5^0 w_0^5) \equiv \varepsilon_{05} = +1,$$

$$SO^{\downarrow}(2, 4): \det(w)=+1, \quad \varepsilon_{05}=-1,$$

$$O_-^{\uparrow}(2, 4): \det(w)=-1, \quad \varepsilon_{05}=+1,$$

$$O_-^{\downarrow}(2, 4): \det(w)=-1, \quad \varepsilon_{05}=-1.$$

From now on we denote by $L_6[\Lambda]$ the subgroup of $O(2, 4)$ which consists of the elements $w=\Lambda\oplus 1_{45}$, where $\Lambda=(w_k^i, i, k=0, 1, 2, 3)$ is an element of the homogeneous Lorentz group and 1_{45} the unity transformation in the (4, 5)-plane. We have

$$L_{6+}^{\uparrow} \subset SO^{\uparrow}(2, 4), \quad L_{6+}^{\downarrow} \subset SO^{\downarrow}(2, 4), \quad \text{etc.}$$

All maximal compact subgroup of $SO^{\uparrow}(2, 4)$ are conjugate (see [11], p. 218) under inner automorphisms to the group $SO(2)\times SO(4)$. Furthermore, the indentity component $SO^{\uparrow}(2, 4)$ is homeomorphic to the space (see [11], p. 345)

$$SO(2)\times SO(4)\times \boldsymbol{R}^8 .$$

As $SO(2)$ is infinitely and $SO(4)$ twofold connected, the group $SO^{\uparrow}(2, 4)$ is infinitely connected, too.

6.1. *The manifold Q_0^5 as a homogeneous space*

The group $SO^{\uparrow}(2, 4)$ acts transitively on Q_0^5. Let $q=(\eta^\mu)$ be any point of Q_0^5. If $SO(2)$ is the subgroup of $SO^{\uparrow}(2, 4)$ acting on (η^0, η^5) and $SO(4)$ the corresponding subgroup acting on $(\eta^1, \ldots \eta^4)$, then there exists an element $g \in SO(2)\oplus SO(4)$ which maps the point q into the point $(\hat{\eta}^0, \hat{\eta}^1, 0, 0, 0, 0)$, $\hat{\eta}^0=\hat{\eta}^1>0$. Furthermore, there is an element $\mathring{g}$ of the 1-parameter special Lorentz group, acting in the (0, 1)-plane, which maps the point $(\hat{\eta}^0, \hat{\eta}^1, 0, 0, 0, 0)$ into the point $\mathring{q}=(1, 1, 0, 0, 0, 0)$. If q_1 and q_2 are any two points of Q_0^5, then $g_2^{-1}\,\mathring{g}_2^{-1}\,\mathring{g}_1 g_1$ maps q_1 into q_2.

The isotropy group of the point $(\eta^j=0, \kappa=1, \lambda=0)\in Q_0^5$ is the subgroup $L_{6+}^{\uparrow}[\Lambda]\otimes SC_4[c]$ of $SO^{\uparrow}(2, 4)$. Thus, we have the following homeomorphisms:

$$\begin{aligned} Q_0^5 &\sim SO^{\uparrow}(2, 4)/\{L_{6+}^{\uparrow}[\Lambda]\otimes SC_4[c]\}, \\ &\sim SO^{\uparrow}(2, 4)/\{L_{6+}^{\uparrow}[\Lambda]\otimes T_4[a]\}. \end{aligned} \tag{57}$$

Since a homogeneous space is the base space of a principle fibre bundle ([19], and [13], p. 71) with the isotropy group as the structure group, we can represent the manifold Q_0^5

as a homogeneous space in terms of a reduced bundle, where the reduced structure group in our case is given by the maximal compact subgroup $1_0 \oplus SO(3) \oplus 1_{45}$ of $L^{\uparrow}_{6+}[A] \otimes T_4[a]$, where 1_0 is the identity transformation on the η^0-axis and $SO(3)$ the rotation group acting on (η^1, η^2, η^3).

In order to construct such a reduced bundle let us consider the group $A^+(6) \times SO(2) \oplus \oplus SO(4)$. It is easy to see that this group acts transitively on the manifold Q_0^5: as discussed above, for every point $q=(\eta^0, \eta^1, \eta^2, \eta^3, \eta^4, \eta^5) \in Q_0^5$ there exists an element $g \in SO(2) \oplus \oplus SO(4)$ which maps q onto the point $(\hat{\eta}^0, \hat{\eta}^1, 0, 0, 0, 0)$, $\hat{\eta}^0 = \hat{\eta}^1 > 0$. Multiplication by the factor $1/\hat{\eta}^0$, which is an element of $A^+(6)$, transforms the point $(\hat{\eta}^0, \hat{\eta}^1, 0, 0, 0, 0)$ into the point $(1, 1, 0, 0, 0, 0)$. The same reasoning as in case of the group $SO^{\uparrow}(2, 4)$ then shows that $A^+(6) \times SO(2) \oplus SO(4)$ acts transitively on Q_0^5, too. The isotropy group of the point $(\eta^j=0,\ \kappa=1,\ \lambda=0) \in Q_0^5$ is the subgroup $1_0 \oplus SO(3) \oplus 1_{45}$ of the group $A^+(6) \times SO(2) \oplus \oplus SO(4)$. Then we have the following homeomorphism:

$$Q_0^5 \sim [A^+(6) \times SO(2) \oplus SO(4)]/[1_0 \oplus SO(3) \oplus 1_{45}]. \tag{58}$$

If we further denote by π^1 the canonical mapping

$$\pi^1:\ SO^{\uparrow}(2, 4) \to SO^{\uparrow}(2, 4)/\{L^{\uparrow}_{6+}[A] \otimes T_4[a]\} \sim Q_0^5$$

and by π^2 the canonical mapping

$$\pi^2:\ A^+(6) \times SO(2) \oplus SO(4) \to [A^+(6) \times SO(2) \oplus SO(4)]/\{1_0 \oplus SO(3) \oplus 1_{45}\} \sim Q_0^5,$$

then we want so show that the principal fibre bundle $(A^+(6) \times SO(2) \oplus SO(4),\ \pi^2,\ Q_0^5)$ with structure group $1_0 \oplus SO(3) \oplus 1_{45}$ is a reduction of the principal fibre bundle $(SO^{\uparrow}(2, 4),\ \pi^1,\ Q_0^5)$ with structure group $L^{\uparrow}_{6+}[A] \otimes T_4[a]$. This can be seen as follows: It is obvious that $1_0 \oplus SO(3) \oplus 1_{45}$ is a closed subgroup of the group $L^{\uparrow}_{6+}[A] \otimes T_4[a]$. Next we define the following mapping $\hat{\alpha}$ of the bundle space $A^+(5) \times SO(2) \oplus SO(4)$ into the bundle space $SO^{\uparrow}(2, 4)$:

$$\hat{\alpha}:\ \alpha \cdot g \mapsto D_1(\alpha) \cdot g, \tag{59}$$

where $\alpha \in A^+(6)$, $g \in SO(2) \oplus SO(4)$ and $D_1(\alpha)$ is the dilatation, characterized by the parameter $\rho = \alpha$ as defined in (22).

The mapping $\hat{\alpha}$ in (59) has the following properties:

(a) $\hat{\alpha}$ is injective,

(b) $\hat{\alpha}$ is a homeomorphism of $A^+(6) \times SO(2) \oplus SO(4)$ onto the closed subset $\hat{\alpha}(A^+(6) \times \times SO(2) \oplus SO(4)) \subset SO^{\uparrow}(2, 4)$,

(c) $\hat{\alpha}(s_1 \cdot s_2) = \hat{\alpha}(s_1) \cdot s_2$ for all $s_1 \in A^+(6) \times SO(2) \oplus SO(4)$ and $s_2 \in 1_0 \oplus SO(3) \oplus 1_{45}$.

The properties (a), (b), (c) of the mapping (59) prove immediately ([13], p. 71) that the fibre bundle $(A^+(6) \times SO(2) \oplus SO(4),\ \pi^2,\ Q_0^5)$ with structure group $1_0 \oplus SO(3) \oplus 1_{45}$ is a reduction of the fibre bundle $(SO^{\uparrow}(2, 4),\ \pi^2,\ Q_0^5)$ with structure group $L^{\uparrow}_{6+}[A] \otimes T_4[a]$.

6.2. *The manifold M_c^4 as a homogeneous space*

Very similar consideration to those given above show that the subgroup $SO(2)\times SO(4)$ of $SO^{\uparrow}(2, 4)$ acts transitively on M_c^4 and the isotropy group of the point $[(\eta^j=0,\ \kappa=0,\ \lambda\neq 0)]$ with respect to $SO^{\uparrow}(2, 4)$ is the group $((L_+^{\uparrow}[\Lambda]\times D_1[\rho])\otimes T_4[a])\times Z_2(6)$. Thus we have

$$M_c^4 \sim SO^{\uparrow}(2, 4)/\{[(L_{6+}^{\uparrow}\times D_1)\otimes T_4]\times Z_2(6)\}. \tag{60}$$

The maximal compact subgroup of the above structure group is the group $(1_0\oplus SO(3)\oplus \oplus 1_{45})\times Z_2(6)$. Therefore we get for the reduced bundle

$$M_c^4 \sim (SO(2)\oplus SO(4))/[(1_0\oplus SO(3)\oplus 1_{45})\times Z_2(6)]. \tag{61}$$

7. Properties of the tangent spaces $T_q(K^5)$, $T_{[q]}(M_c^4)$ and their duals

The structure of the tangent spaces of the manifolds $K^5(E, b)$ and M_c^4 or Q_0^5 and Q_0^5/A, respectively, is of considerable physical significance as far as the notions of causality are concerned, particularly in the context of local quantum field theory. One may ask whether it is possible to define a nontrivial conformally invariant "local" field theory even if the structure of such a theory is much poorer than that of the real world. In order to prepare the ground for the analysis of such a theory, we first have to clarify some geometrical questions. Their implication for the notion of causality will be discussed in the next section. The problem of the existence or nonexistence of a conformally invariant local quantum field theory will not be investigated in this paper.

7.1. *The tangent spaces $T_q(Q_0^5)$ and their duals $T_q^*(Q_0^5)$*

It is often convenient to consider the manifold $Q_0^5=\bigcup_\alpha \hat{N}_{(\alpha)}^5 \sim K^5(E, b)$ *via* the inclusion mapping as a submanifold of the manifold $\mathring{R}_2^6$ which is the space $R^6-\{0\}$ with the bilinear form

$$k(\overset{1}{\tilde{q}}, \overset{2}{\tilde{q}})=\eta^\mu(1)\,\eta_\mu(2). \tag{62}$$

(We write $\tilde{q}$ for the elements of $\mathring{R}_2^6$ and $q=\tilde{q}$ if $\tilde{q}\in Q_0^5$.) As the mapping

$$\mathring{R}_2^6\to R:\ \xi=\eta^\mu\eta_\mu \tag{63}$$

has maximal rank one—because $\partial\xi/\partial\eta^\mu\neq 0$—the set $Q_0^5=\xi^{-1}(0)$ is a closed submanifold of $\mathring{R}_2^6$. The inclusion i: $Q_0^5\to\mathring{R}_2^6$ may also be characterized as the subset of all isotropic vectors $\tilde{q}\in\mathring{R}_2^6$:

$$Q_0^5=\{\tilde{q}\in\mathring{R}_2^6,\ k(\tilde{q}, \tilde{q})=0\}.$$

Next we shall discuss briefly several coordinate systems on $\mathring{R}_2^6$ and Q_0^5 which are useful in different contexts.

Whereas the coordinates η^μ, $\mu=0, \ldots, 5$, in $\mathring{R}_2^6$ are often convenient in connection with the action of the group $O(2,4)$, it is generally more appropriate to use the coordinates κ, λ, $\tilde{\kappa}$ $\tilde{\lambda}$, η^2, η^3 when dealing with local properties on the charts $\hat{N}_{(\alpha)}^5$.

Another useful local coordinate system is the following: According to equation (63) the submanifold Q_0^5 can be characterized by $\xi=0$. It is a standard result that one can always introduce additional five coordinates $\xi^\beta=\xi^\beta(\eta)$, $\beta=1, \ldots, 5$ in a neighbourhood $\subset \mathring{R}_2^6$ of $q \in Q_0^5 \subset \mathring{R}_2^6$, such that $(\xi=0, \xi^\beta)$ is a local coordinate system on Q_0^5 in a neighbourhood of $q \in Q_0^5$ and that the Jacobian of the corresponding coordinate transformation is nonvanishing. In a 6-dimensional neighbourhood of $q \in \hat{N}_{(\kappa)}^5$, in which $\kappa \neq 0$, for instance, it is convenient to replace the variable λ by $\xi_{(\kappa)}=\eta^j\eta_j-\kappa\lambda$, and choose $(\xi^\beta)=(\eta^j, \kappa)$. In this case the Jacobian has the value $-\kappa \neq 0$.

The following example illustrates the application of these coordinates. If $f(q)$ is a C^∞-function on Q_0^5 having the form $f_{(\kappa)}(\eta^j, \kappa)$ on $\hat{N}_{(\kappa)}^5$, then there is always a C^∞-function $\tilde{f}$ on $\mathring{R}_2^6$ with the property

$$f_{(\kappa)}(\eta^j, \kappa)=\tilde{f}(\eta^j, \kappa, \xi_{(\kappa)}=0) \text{ on } \hat{N}_{(\kappa)}^5 .$$

For instance, if $\Theta(\xi_{(\kappa)})$ is the C^∞-function which is equal to 1 within the compact closure of a neighbourhood U of $\xi_{(\kappa)}=0$ and equal to zero outside an open neighbourhood containing $\bar{U}\,(\xi_{(\kappa)}=0)$, then

$$\tilde{f}(\eta^j, \kappa, \xi_{(\kappa)})=\Theta(\xi_{(\kappa)}) f_{(\kappa)}(\eta^j, \kappa)$$

is such a function.

Next we ask for the condition for a tangent vector

$$\tilde{a}^\mu(\tilde{q})\,\partial_\mu \in T_{\tilde{q}}(\mathring{R}_2^6), \qquad \partial_\mu \equiv \partial/\partial\eta^\mu,$$

at $\tilde{q}=q \in Q_0^5$ to be a tangent vector $\in T_q(Q_0^5)$. Now, any smooth curve $C(t)$, $0<t<1$ in Q_0^5 must satisfy $(d\xi/dt)_{\xi=0}=0$, because ξ is constant on Q_0^5. Switching from the coordinates η^μ to (ξ, ξ^β), we see, according to equation (17), that $d\xi/dt=0$ implies

$$\eta_\mu(q)\,\tilde{a}^\mu(q)=0, \quad q \in Q_0^5. \tag{64}$$

On the other hand, the relation

$$(d\xi/dt)_{t=t_0}=\eta_\mu(t_0)\,\tilde{a}^\mu=0, \quad \tilde{a}^\mu=\text{const},$$

together with $\xi\ (t=t_0)=0$, implies that the tangent vector $\tilde{a}^\mu\partial\mu$ has no component in the ξ-direction. Therefore equation (64) is also sufficient for $\tilde{a}^\mu\partial\mu$ to be an element of $T_q(Q_0^5)$.

Another way of looking at the condition (64) is as follows: The function $\xi=\eta^\mu\eta_\mu$ is constant on Q_0^5. Thus we have

$$0=d\xi=2\eta_\mu\, d\eta^\mu \text{ on } Q_0^5, \tag{65}$$

where $d\eta^\mu \in T_q^*(\mathring{R}_2^6)$. If $\tilde{a}^\mu(\tilde{q})\partial_\mu \in T_{\tilde{q}}(\mathring{R}_2^6)$, then (65) implies (64).

If we write

$$\tilde{a}_{(\kappa)}=\tilde{a}^4-\tilde{a}^5, \qquad \tilde{a}_{(\lambda)}=\tilde{a}^4+\tilde{a}^5,$$

the tangent vectors in $T_{\tilde{q}}(\mathring{R}_2^6)$ take the form

$$\tilde{a}^j(q)\,\partial_j+\tilde{a}_{(\kappa)}\,\partial/\partial\kappa+\tilde{a}_{(\lambda)}\,\partial/\partial\lambda.$$

If $\hat{N}^5_{(\kappa)}\subset V_{(\kappa)}=\{\tilde{q}\in\mathring{R}^6_2,\ \kappa(\tilde{q})\neq 0\}$, then we have

$$\tilde{a}^\mu\partial_\mu\mapsto X^{(\kappa)}_{\tilde{q}}=\tilde{a}^j\partial_j+\tilde{a}_{(\kappa)}\,\partial/\partial\kappa+\tilde{a}_{(\xi)}\,\partial/\partial\xi_{(\kappa)},$$
$$\tilde{a}_{(\xi)}=2\eta^j\tilde{a}_j-\kappa\tilde{a}_{(\lambda)}-\lambda\tilde{a}_{(\kappa)} \tag{66}$$

acting on functions $f(\eta^j,\kappa,\xi_{(\kappa)})$.

Let us assume that X_q in (66) is a vector field, i.e. the functions $\tilde{a}^\mu(q)$ are C^∞. Then the vector field $X^{(\kappa)}_{\tilde{q}}$ induces the vector field

$$a^j(q)\,\partial_j+a_{(\kappa)}(q)\,\partial/\partial_\kappa, \qquad q\in\hat{N}^5_{(\kappa)}, \tag{67}$$
$$a^j(q)=\tilde{a}^j(\eta^j,\kappa,\xi_{(\kappa)}=0),$$
$$a_{(\kappa)}(q)=\tilde{a}_{(\kappa)}(\eta^j,\kappa,\xi_{(\kappa)}=0),$$
$$a_{(\lambda)}(q)=\frac{1}{\kappa}\left[2a^j\eta_j-\lambda a_{(\kappa)}\right],$$

on Q^5_0. On the other hand, any vector field on Q^5_0 induces a vector field on $\mathring{R}^6_2$.

The physical meaning of the quantity $a_{(\kappa)}$ can be explained as follows. If $C(t)$ is a curve in Q^5_0, then $d\kappa/dt=a_{(\kappa)}(q)$, i.e. $a_{(\kappa)}(q)$ gives the change of the rod $b(t)=(1/\kappa(t))\hat{b}$ along the curve $C(t)$.

Without referring to any special chart, we can state that the set

$$\{\eta_\mu\,\partial_\nu-\eta_\nu\,\partial_\mu,\ (\eta)\in Q^5_0\}$$

spans $T_q(Q^5_0)$ at each $q(\eta)\in Q^5_0$, because the tangent vectors $\eta_\mu\partial_\nu-\eta_\nu\partial_\mu$ satisfy the condition (64) automatically on Q^5_0.

In the same sense the set

$$\{\eta^\nu d\eta^\mu-\eta^\mu d\eta^\nu,\ (\eta)\in Q^5_0\}$$

spans $T^*_q(Q^5_0)$.

On $T_{\tilde{q}}(\mathring{R}^6_2)$ we can introduce the metric

$$h_{\tilde{q}}=d\eta^\mu\otimes d\eta_\mu \tag{68}$$

such that

$$h_{\tilde{q}}(\overset{1}{\tilde{a}}{}^\mu(\tilde{q})\,\partial_\mu,\ \overset{2}{\tilde{a}}{}^\mu(\tilde{q})\,\partial_\mu)=\overset{1}{\tilde{a}}{}^\mu(\tilde{q})\,\overset{2}{\tilde{a}}_\mu(\tilde{q}).$$

When restricted to $T(Q_0^5)$, the metric h_q is degenerate. On $\hat{N}^5_{(\kappa)}$, for instance, we have

$$h_q^{(\kappa)} = d\eta^j \otimes d\eta_j + \frac{\lambda}{\kappa}\, d\kappa \otimes d\kappa - 2\cdot\frac{\eta_j}{\kappa}\, d\kappa \otimes d\eta^j .$$

The 5×5 determinant of this bilinear form is equal to $\eta^\mu \eta_\mu / \kappa^2$, i.e. it vanishes on $\hat{N}^5_{(\kappa)}$.

As the expressions $a^\mu(q)\, a_\mu(q)$ are invariants under the group $O(2,4)$, we can define in an $O(2,4)$-invariant way the notion of "time-like", "space-like" and "light-like" tangent vectors $\in T(Q_0^5)$ by sign $(a^\mu a_\mu) = +1$, -1 and 0, respectively.

The proof of the following important theorem may be found in Appendix A:

THEOREM I. *If* $a^\mu(q)\partial_\mu \in T_q(Q_0^5)$, *i.e. if* $\eta^\mu a_\mu(q) = 0$, *then* $\operatorname{sgn}[\eta^0 a^5(q) - \eta^5 a^0(q)]$ *is invariant under the group* $O^\uparrow(2,4)$ *iff* $a^\mu(q)\, a_\mu(q) \geq 0$.

The importance of this theorem will become clear in the next section.

Contrary to the situation in the Minkowski space, we cannot in general identify the manifold Q_0^5 with one of its tangent spaces. However, there is the following important special case: Let $\mathring{q}$ be an arbitrary but fixed point of Q_0^5. Denote by

$$\hat{L}(\mathring{q}) = \{q \in Q_0^5,\; k(\mathring{q}, q) \equiv \mathring{\eta}^\mu(\mathring{q})\, \eta_\mu(q; \mathring{q}) = 0\} \tag{69}$$

the set of all points $q = q(\mathring{q}) \in Q_0^5$ with coordinates $\eta^\mu(q; \mathring{q})$ which are light-like to $\mathring{q}$.

Furthermore, define

$$\hat{H}(\mathring{q}) = \{a^\mu(\mathring{q})\,\partial_\mu \in T^0_{\mathring{q}}(Q_0^5),\; a^\mu(\mathring{q})\, a_\mu(\mathring{q}) = 0,\; a^\mu(\mathring{q}) + \mathring{\eta}^\mu \neq 0\}, \tag{70}$$

i.e. $\hat{H}(\mathring{q})$ is the set of all light-like vectors contained in $T_{\mathring{q}}(Q_0^5) - \{-\mathring{\eta}^\mu \partial_\mu\}$. Then there is a bijective mapping

$$\chi:\; \hat{H}(\mathring{q}) \to \hat{L}(\mathring{q}),$$

defined by

$$\chi:\; a^\mu(\mathring{q})\,\partial_\mu \to \eta^\mu(q; \mathring{q}) = \mathring{\eta}^\mu + a^\mu(\mathring{q}) \in \hat{L}(\mathring{q}), \tag{71}$$

$$\chi^{-1}:\; \eta^\mu(q; \mathring{q}) \in \hat{L}(\mathring{q}), \quad \eta^\mu \to a^\mu(\mathring{q})\,\partial_\mu, \quad a^\mu(\mathring{q}) = \eta^\mu(q; \mathring{q}) - \mathring{\eta}^\mu .$$

Note that $\hat{H}(\mathring{q})$ and $\hat{L}(\mathring{q})$ are not linear spaces, for if $\dot{a}^\mu(\mathring{q})\partial_\mu$, $\ddot{a}^\mu(\mathring{q})\partial_\mu \in \hat{H}(\mathring{q})$, their sum $(\dot{a}^\mu + \ddot{a}^\mu)\partial_\mu$ is in general not an element of $\hat{H}(\mathring{q})$. Note further that $\mathring{q} \in \hat{L}(q)$ if $q \in \hat{L}(\mathring{q})$.

Let us briefly discuss the structure of the space $\hat{L}(\mathring{q})$. We find the topological equivalence

$$\hat{L}(\mathring{q}) - \{\mathring{q}\} \sim \boldsymbol{R} \times S^2 \times \boldsymbol{R}_0 . \tag{72}$$

This may be seen as follows: Without loss of generality we may choose $\mathring{\eta}^j = 0$, $\mathring{\lambda} = 0$, $\mathring{\kappa} \neq 0$. Then the coordinates of the points $q \in \hat{L}(\mathring{q})$ must satisfy the equations

$$(\eta^0)^2 + (\eta^5)^2 = (\eta^1)^2 + \ldots + (\eta^4)^2, \quad \eta^5 = -\eta^4,$$

or

$$(\eta^0)^2 = (\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2, \quad \kappa \in \boldsymbol{R},\; \lambda = 0 .$$

This immediately implies the relation (72). From the topological structures alone of the light cone $\hat{L}(\mathring{q})$ in the manifold Q_0^5 we see immediately that it is not possible to introduce a global time-ordering on the "light cone" $\tilde{L}([\mathring{q}])$ in the manifold $M_c^4 = Q_0^5/R$: a projection of $\hat{L}(\mathring{q}) - \{\mathring{q}\}$ into the manifold M_c^4 gives

$$L([\mathring{q}]) - \{[\mathring{q}]\} \sim \boldsymbol{R} \times S^2 .$$

For the analogous light cone $L(y) := \{x \in M^4 \colon (y-x)^2 = 0\}$ in Minkowski space, on the other hand, we get

$$L(y) - \{y\} \sim \boldsymbol{R}_0 \times S^2 .$$

We see that the factor $\boldsymbol{R}_0$ allows to introduce a global time-ordering on $L(y)$, whereas in the case of the space $\tilde{L}([\mathring{q}]) - \{[\mathring{q}]\}$ the factor $\boldsymbol{R} = \boldsymbol{R}_0 \cup \{0\}$ does not, because $\boldsymbol{R}$ is *simply* connected, but $\boldsymbol{R}_0$ is not and its two pieces are past and future relative to the origin. This argument tells us that on the manifold M_c^4 we cannot expect a global time-ordering. But we shall show that a local causal structure can be introduced on M_c^4.

However, it is possible to introduce a time-ordering on covering spaces of M_c^4, for instance, $K^5(E, b)/A^-$ is a twofold covering of M_c^4 and, because of $A^+ \sim \boldsymbol{R}$, we have

$$(\hat{L}(\mathring{q}) - \{\mathring{q}\})/A^+ \sim \boldsymbol{R}_0 \times S^2 .$$

It is even possible to show that the time-ordering can be made an $O^{\uparrow}(2, 4)$-invariant one. This follows from the results in Appendix A.

7.2. *The tangent spaces $T_{[q]}(\boldsymbol{R}_{1c}^4)$ and their duals $T_{[q]}^*(\boldsymbol{R}_{1c}^4)$*

The projection $\hat{\pi}$: $Q_0^5 \to Q_0^5/A \equiv \boldsymbol{R}_{1c}^4 \sim M_c^4$ induces the following mapping of tangent spaces

$$\hat{\pi}_* \colon \; T_q(Q_0^5) \to T_{\hat{\pi}(q)}(\boldsymbol{R}_{1c}^4) .$$

On $N_{(\kappa)}^5$, for instance, we have

$$\hat{\pi} \colon \; q = (\eta^j, \kappa) \mapsto x = (x^j = \eta^j/\kappa) ;$$

$$\hat{\pi}_* \colon \; a^j(q)\,\partial_j + a_{(\kappa)}\,\partial/\partial_\kappa \to \hat{\pi}_*(X_q^{(\kappa)}) ,$$

where

$$(\hat{\pi}_* X_q^{(\kappa)}) f(\hat{\pi}(q)) = X_q^{(\kappa)} f[x(q)] = \frac{1}{\kappa}\left[a^j(q)\,\partial/\partial x^j f(x) - a_{(\kappa)}(q)\, x^j \partial/\partial x^{j}\, f^{(x)}\right] . \tag{73}$$

Thus for a vector field $Y = s^\mu(q)\,\partial_\mu$, $\hat{\pi}_* Y$ is a vector field on $\boldsymbol{R}_{1c}^4$ if the functions $s^\mu(q)$ are homogeneous of degree $+1$, i.e. if Y is A-invariant. In this case we get

$$(\hat{\pi}_* Y^{(\kappa)})_{\hat{\pi}(q)} = S^j(\hat{\pi}(q))\,\partial/\partial x^j , \quad S^j(\hat{\pi}(q)) = s^j(x) - x^j s_{(\kappa)}(x) , \quad s^j(x) = s^j(q/\kappa), \text{ etc.}$$

In particular, we have

$$\hat{\pi}_*(\eta^j \partial_j + \kappa\,\partial/\partial\kappa) = 0 \text{ on } \hat{N}_{(\kappa)}^5 .$$

Similar relations as above hold on the other charts $\hat{N}^5_{(\alpha)}$ and their projections. Independently of any chart we have

$$\hat{\pi}_*(\eta^\mu \partial_\mu) = 0 ,$$

because $\eta^\mu \partial_\mu$ is the vector field induced by A on Q^5_0 and the projection of this field has to vanish because A acts as the identity transformation on R^4_{1c}.

Let us give a more explicit description of the tangent spaces of the manifold M^4_c at a point $\hat{\pi}(\mathring{q}) = [\mathring{q}]$. The symbol $[\mathring{q}]$ denotes the class of all points $q \in Q^5_0$ such that $\hat{\pi}(q) = \hat{\pi}(\mathring{q})$. The definition of the mapping $\hat{\pi}$ shows that $[\mathring{q}] = \{q \in Q^5_0 \colon \eta^\mu = \rho \mathring{\eta}^\mu,\ \rho \in R_0\}$.

Let $T_q(Q^5_0)$ be the tangent space of Q^5_0 at the point q. Then we consider the following set $T(Q^5_0)$

$$T(Q^5_0) := \bigcup_{q \in Q^5_0} T_q(Q^5_0) , \tag{74}$$

the so-called *tangent bundle* of the manifold Q^5_0. The differentiable structure on Q^5_0 induces in a natural way ([19]) a differentiable structure on this manifold. We shall denote the elements of $T(Q^5_0)$ by the ordered pairs $(q, a^\mu \partial_\mu)$, where $q \in Q^5_0$ and $a^\mu \partial_\mu \in T_q(Q^5_0)$.

Next we introduce the following equivalence relation $\underset{\text{eq}}{\sim}$ in the manifold $T(Q^5_0)$:

$$(q, a^\mu \partial_\mu) \underset{\text{eq}}{\sim} (q', a^{\mu'} \partial_\mu) \quad \exists \rho \in R_0 \text{ and } \sigma \in R \text{ such that } \eta^{\mu'} = \rho \eta^\mu \text{ and } a^{\mu'} = \rho a^\mu + \sigma \eta^\mu . \tag{75}$$

One can easily show that this is indeed an equivalence relation. We denote by $\tilde{T}$ the quotient space

$$\tilde{T} := T(Q^5_0) / \underset{\text{eq}}{\sim}$$

which is induced by (75), and by $[(q, a^\mu \partial_\mu)]$ its elements.

We want to show that the space $\tilde{T}$ can be identified with the tangent bundle $T(M^4_c)$ of the manifold M^4_c.

For this reason we define the mapping $\tilde{\pi} \colon \tilde{T} \to T(M^4_c)$ by

$$\tilde{\pi} \colon [(q, a^\mu \partial_\mu)] \mapsto (\hat{\pi}(q), \hat{\pi}_*(a^\mu \partial_\mu)) . \tag{76}$$

The right-hand side is an element of $T(M^4_c)$ because $\hat{\pi}(q) \in M^4_c$ and $\hat{\pi}_*(a^\mu \partial_\mu) \in T_{[q]}(M^4_c)$.

We must still show that the mapping $\tilde{\pi}$ in (76) is indeed a well defined object on $\tilde{T}$, i.e. that it does not depend on the choice of the representative of the class $[(q, a^\mu \partial_\mu)]$.

So let $(q, a^\mu \partial_\mu)$ and $(q', a^{\mu'} \partial_\mu)$ be two representatives of the class $[(q, a^\mu \partial_\mu)]$. Then by definition (75) there exist two numbers $\rho \in R_0$ and $\sigma \in R$ such that $\eta^{\mu'} = \rho \eta^\mu$ and $a^{\mu'} = \rho a^\mu + \sigma \eta^\mu$. Therefore we have

$$(\hat{\pi}(q'), \hat{\pi}_*(a^{\mu'} \partial_\mu)) = ((\hat{\pi}(\rho, q), \hat{\pi}_*(\rho a^\mu + \sigma \eta^\mu)) = (\hat{\pi}(q), \hat{\pi}_*(\rho a^\mu + \sigma \eta^\mu)) .$$

Without loss of generality we can assume $q \in N^5_\kappa$; then also $q' \in N^5_\kappa$. On N^5_κ the mapping $\hat{\pi}_*$ was given in (73). Therefore $\hat{\pi}_*(a^\mu \partial_\mu) = A^j \partial_j$, where $A^j \partial_j \in T_{[q]}(M^4_c)$ and $A^j = \dfrac{a^j}{\kappa} - \dfrac{\eta^j}{\kappa} \dfrac{a^\kappa}{\kappa}$.

For $\hat{\pi}_*(a^{\mu'}\partial_\mu)$, on the other hand, we get:

$$\hat{\pi}_*(a^{\mu'}\partial_\mu)=A'^j\partial_j\in T_{[q]}(M_c^4)$$

with

$$A'^j=\frac{a^{j'}}{\kappa'}-\frac{\eta^{j'}}{\kappa'}\frac{a^{\kappa'}}{\kappa'}$$

or

$$A'^j=\frac{\rho a^j+\sigma\eta^j}{\rho\kappa}-\frac{\rho\eta^j}{\rho\kappa}\frac{\rho a_\kappa+\sigma\kappa}{\rho\kappa}=a^j/\kappa-\eta^j/\kappa\cdot a_\kappa/\kappa=A^j.$$

Thus $(\hat{\pi}(q'),\hat{\pi}_*(a^{\mu'}\partial_\mu))=(\hat{\pi}(q),\hat{\pi}_*(a^\mu\partial_\mu))$ and therefore the mapping $\tilde{\pi}$ is well defined. Since $\tilde{\pi}$ is also invertible, it is even a diffeomorphism. In what follows we always identity $\tilde{T}$ with the tangent bundle $T(M_c^4)$. The tangent space $T_{[q]}(M_c^4)$ of the manifold M_c^4 at a point $[q]$ can now be defined also in the following way:

Let $q\in Q_0^5$ be a representative of the class $[q]$. Then consider the set $\tilde{T}_{[q]}(M_c^4)$ of all classes $[(q, a^\mu\partial_\mu)]\in T(M_c^4)$, where $[q]$ is kept fixed. This set $\tilde{T}_{[q]}(M_c^4)$ is diffeomorphic to the tangent space $T_{[q]}(M_c^4)$ of the manifold M_c^4 at the point $[q]$. We shall identify these two spaces, too.

Besides the projection $\hat{\pi}_*$ we have the induced mapping

$$\hat{\pi}^*:\ T^*_{\hat{\pi}(q)}(R^4_{1c})\to T^*_q(Q^5_0)$$

of cotangent spaces. On $T^*_{\hat{\pi}(q)}(R_1^4(\kappa))$ for instance, we have

$$\hat{\pi}^*(dx^j)=\frac{1}{\kappa}\left(d\eta^j-\frac{\eta^j}{\kappa}\,d\kappa\right),$$

$$d\lambda=\frac{1}{\kappa}(-\lambda d\kappa+2\eta_j\,d\eta^j). \tag{77}$$

From this we get for $X_q^{(\kappa)}=a^j(q)\partial_j+a_{(\kappa)}(q)\partial/\partial\kappa$, with

$$\hat{\pi}_*(X_q^{(\kappa)})=A^j(x)\partial/\partial x^j;\qquad a_{(\lambda)}(x)=2x_j a^j(x)-x^2a_{(\kappa)}(x),$$

the scalar product

$$\begin{aligned}\tilde{g}_{\hat{\pi}(q)}(\hat{\pi}_*(X_q^{(\kappa)}),\hat{\pi}_*(X_q^{(\kappa)}))&=dx^j(\hat{\pi}_*(X_q^{(\kappa)}))\otimes dx_j(\hat{\pi}_*(X_q^{(\kappa)}))\\&=a^j(x)a_j(x)-a_{(\kappa)}(x)a_{(\lambda)}(x)\\&=\frac{1}{\kappa^2}d\eta^\mu(X_q^{(\kappa)})\otimes d\eta_\mu(X_q^{(\kappa)})\\&=\frac{1}{\kappa^2}h_q(X_q^{(\kappa)},X_q^{(\kappa)}).\end{aligned} \tag{78}$$

Thus we can express the scalar product $\tilde{g}$ on $T_{\hat{\pi}(q)}(R_1^4(\kappa))$ in terms of the (degenerate) scalar product on $T_q(\hat{N}^5_{(\kappa)})$ and the notions of "space-like", "time-like" and "light-like" for the corresponding elements in these spaces are the same, because $\kappa^2 > 0$. Note that this is the case, even if $\hat{\pi}_*(X_q^{(\kappa)})$ is not a vector field on $R_1^4(\kappa)$. Our previous definition of "time-like" tangent vectors etc., is therefore a generalization of the usual one.

On $R_1^4(\lambda)$ we have

$$\hat{\pi}^*(dy^j \otimes dy_j) = \frac{1}{\lambda^2} d\eta^\nu \otimes d\eta_\nu . \tag{79}$$

On $R_1^4(\kappa) \cap R_1^4(\lambda)$ the expression (79) does not give the same numerical value for the length of a given tangent vector as the expression (78), because the "scale" or "gauge" factors κ and λ are in general different. However, the notions of "time-like" vectors etc., are invariant ones.

The description of the tangent spaces $T_{[q]}(M_c^4)$ and the tangent bundle $T(M_c^4)$ above enables us now to introduce the concept of a conformally invariant "local causal structure" on the manifold M_c^4. The role which tangent bundles and, more generally, fibre bundles play in physics was already pointed out by Trautman [27]. Before doing this we should make clear what we mean in the case of Minkowski space by a causal structure and then generalize this concept to the manifold M_c^4.

8. The local causal structure on M_c^4

8.1. *The causal structure on Minkowski space M_4*

Let two points x and y be given in Minkowski space M_4. Then there exist two functions $c\colon M_4 \times M_4 \to \boldsymbol{R}$ and $t\colon M_4 \times M_4 \to \boldsymbol{R}$ which describe the causal structure ([33]) of Minkowski space:

$$\begin{aligned} c\colon\ & M_4 \times M_4 \to \boldsymbol{R}, \\ & (x, y) \to \varepsilon(g^{ik}(y_i - x_i)(y_k - x_k)); \end{aligned} \tag{80}$$

$$t\colon\ M_4 \times M_4 \to \boldsymbol{R},$$

$$(x, y) \mapsto \varepsilon(y^0 - x^0), \text{ where } \varepsilon(\alpha) = \begin{cases} 1 & \text{iff} \quad >0, \\ 0 & \text{iff} \quad \alpha = 0, \\ -1 & \text{iff} \quad <0. \end{cases} \tag{81}$$

It is well known that the function c is a Poincaré-invariant mapping, i.e. for all $(x, y) \in M_4 \times \times M_4$ and all $P \in P_{10}(\Lambda, a)$

$$c(Px, Py) = c(x, y).$$

Further, we know that the function $t\colon M_4 \times M_4 \to \boldsymbol{R}$ is invariant under the action of the group

$P_{10}(\Lambda, a)$ for all points $(x, y) \in M_4 \times M_4$ with $c(x, y) \geq 0$, that is, for all light or time-like pairs (x, y),

$$t(Px, Py) = t(x, y) \quad \text{for all } P \in P_{10}(\Lambda, a) \text{ and all } (x, y) \quad \text{with } c(x, y) \geq 0.$$

Let us briefly summarize some more properties of the functions c and t.

(1a) *c is symmetric, i.e.* $c(x, y) = c(y, x)$; (82a)

(1b) *if* $c(x, y) \geq 0$ *and* $c(y, z) \geq 0$, *then it follows that* $c(x, z) \geq 0$,

$$\textit{if } t(x, y) > 0 \textit{ and } t(y, z) > 0; \tag{82b}$$

(2a) *t is antisymmetric, i.e.* $t(x, y) = -t(y, x)$; (83a)

(2b) *if* $t(x, y) \geq 0$ *and* $t(y, z) \geq 0$, *then also* $t(x, z) \geq 0$. (83b)

The two functions c and t define what we understand by a causal structure. In general, the set $M_4 \times M_4$ is interpreted as the direct product of the Minkowski space M_4 with itself. But according to our discussion in Section 2.1, abstracted in the diagram following equation (11), we also can interpret the situation in the following way: Since M_4 is a linear space, we know that the tangent space of M_4 at any point can be mapped isomorphically via the mapping σ in equation (1) onto the space M_4 itself. Therefore, the set M_4 can be looked upon as the product of the space M_4 and one of its tangent spaces at a given point. That is to say, in the case of a vector space V we can identify the direct product $V \times V$ with the tangent bundle $T(V)$ of the space V. In our case this means

$$M_4 \times M_4 \cong T(M_4). \tag{84}$$

The functions c and t defined in (80) and (81) as mappings on $M_4 \times M_4$ now appear as mappings c and t on the tangent bundle $T(M_4)$:

$$c\colon\ T(M_4) \to \boldsymbol{R}, \tag{85}$$

$$t\colon\ T(M_4) \to \boldsymbol{R}, \tag{86}$$

and the causal structure is therefore defined by these two functions on the tangent bundle.

However, by switching from the direct product $M_4 \times M_4$ to the tangent bundle $T(M_4)$ we, clearly, lose the properties of the functions c and t which are closely related to the structure of the direct product $M_4 \times M_4$, i.e. the properties which we have listed in equations (82a)–(83b).

But we know that we can always introduce a local coordinate system on the tangent bundle $T(M)$ of any differentiable manifold M such that the tangent bundle is locally equivalent (via the exponential map) to the direct product of some open set of the manifold M with itself. Then we demand that the functions $\bar{c}$ and $\bar{t}$, which are globally defined on the tangent bundle, have locally, i.e. on $U_\kappa \times U_\kappa \subset M_c^4 \times M_c^4$, exactly all the properties of the functions c and t, especially the properties (82a) to (83b). If we can find on the tangent bundle of a manifold such functions then we say that the manifold itself has a *local causal structure*. As we have seen above, in case of the Minkowski space M_4 the local structure is even a global one, but in general we cannot expect this.

Let us now consider the manifold M_c^4. We shall see in the next paragraph that we can introduce on it a conformally invariant local causal structure in the sense discussed above. This means that there exist globally defined functions $\bar{c}$ and $\bar{t}$ on the tangent bundle $T(M_c^+)$, and for every point $[(q, a^\mu\partial_\mu)] \in T(M_c^+)$ there is a local coordinate system, say (V, Ψ), such that in these local coordinates the functions $\bar{c}$ and $\bar{t}$ have exactly the same properties as the functions c and t in Minkowski space. Moreover, the functions $\bar{c}$ and $\bar{t}$ are conformally invariant.

8.2. *The local causal structure on the manifold M_c^4*

The way we described the tangent bundle $T(M_c^+)$ in Section 7 is of considerable advantage for our purpose: Let us define the following two functions $\bar{c}$ and $\bar{t}$:

$$\bar{c}\colon\ T(M_c^+)\to \boldsymbol{R},$$
$$[(q, a^\mu\partial_\mu)]\mapsto \varepsilon(a^\mu a_\mu), \tag{87}$$

and

$$\bar{t}\colon\ T(M_c^+)\to \boldsymbol{R},$$
$$[(q, a^\mu\partial_\mu)]\mapsto \varepsilon(\eta^0 a^5-\eta^5 a^0). \tag{88}$$

We show that the functions $\bar{c}$ and $\bar{t}$ are well defined everywhere on $T(M_c^+)$. First, take the function $\bar{c}$ and let $(q', a^{\mu'}\partial_\mu)$ be any representative of the class $[(q, a^\mu\partial_\mu)]$. Then, by definition, there exist a $\rho \in \boldsymbol{R}_0$ and a $\sigma \in \boldsymbol{R}$ such that $\eta^{\mu'}=\rho\eta^\mu$ and $a^{\mu'}=\rho a^\mu+\sigma\eta^\mu$. Therefore $\varepsilon(a^{\mu'}a_\mu')=\varepsilon(\rho^2 a_u a^\mu)=\varepsilon(a^\mu a_\mu)$, because $\rho \in \boldsymbol{R}_0$ and $a^\mu\eta_\mu=\eta^\mu\eta_\mu=0$. Thus the definition of $\bar{c}$ does not depend on the choice of the representative of the class $[(q, a^\mu\partial_\mu)]$ and is therefore well defined everywhere on $T(M_c^+)$.

Next, consider the function $\bar{t}$ and take again any representative $(q', a^{\mu'}\partial_\mu)$ of the class $[(q, a^\mu\partial_\mu)]$. The same reasoning as above leads to

$$\varepsilon(\eta^{0'}a^{5'}-\eta^{5'}a^{0'})=\varepsilon\big(\rho^2(\eta^0 a^5-\eta^5 a^0)\big)=\varepsilon(\eta^0 a^5-\eta^5 a^0)$$

and this shows that also $\bar{t}$ is a well defined mapping of $T(M_c^+)$ into the set $\{+1, -1, 0\} \subset \boldsymbol{R}$.

DEFINITION. The point $[(q, a^\mu\partial_\mu)] \in T(M_c^+)$ is called *timelike*, *lightlike* or *spacelike* iff $\bar{c}([(q, a^\mu\partial_\mu)])>0, =0$, or <0, respectively.

It follows from our discussion in Section 7 that these definitions are conformally invariant.

We can now state the following theorem which solves the problem of introducing a local causal structure on M_c^4.

THEOREM II. (a) *The function* $\bar{t}$: $T(M_c^4)\to R$ *is conformally invariant for all lightlike or timelike points, i.e.*

$$\bar{t}([(q, a^\mu\partial_\mu)])=\bar{t}(w[(q, a^\mu\partial_\mu)]) \quad \text{iff} \quad \bar{c}([(q, a^\mu\partial_\mu)])\geqslant 0,$$

where $w\in O^\uparrow(2,4)$ *and* $w[(q, a^\mu\partial_\mu)]:=[(wq, w_* a^\mu\partial_\mu)]$.

(b) *For every point* $[(q, a^\mu\partial_\mu)]\in T(M_c^4)$ *there exists a* $C_{15}^\uparrow$*-equivalent chart* (V,Ψ) *such that* $\bar{c}\circ\Psi^{-1}(z_1,z_2)=c(z_1,z_2)$ *for all* $(z_1,z_2)\in\Psi(V)\subset R^4\times R^4$ *and* $\bar{t}\circ\Psi^{-1}(z_1,z_2)=t(z_1,z_2)$ *for all* $(z_1,z_2)\in\Psi(V)$ *with* $c(z_1,z_2)\geqslant 0$.

Proof: (a) The result follows immediately from the theorems in Appendix A.

(b) Let (U,Φ) be the local chart of $T(M_c^4)$ which is induced ([19]) by the chart $(U_\kappa,\varphi_\kappa)$ of M_c^4, where $U_\kappa=\{[\eta]\in M_c^4: \kappa\neq 0\}$ and φ_κ: $U_\kappa\to M_4$ a homeomorphism, defined by φ_κ: $[\eta]\mapsto\left(\frac{\eta^0}{\kappa},\frac{\eta^1}{\kappa},\frac{\eta^2}{\kappa},\frac{\eta^3}{\kappa}\right)$.

The homeomorphism Φ: $U\to M_4\times M_4$ is defined in the following way:

$$\Phi: [(\eta, a^\mu\partial_\mu)]\mapsto(\varphi_\kappa([\eta]),\quad \sigma_{\varphi_\kappa([\eta])}\circ\hat{\pi}_{\kappa*}(a^\mu\partial_\mu))=(x_1,x_2),$$

where $x_1=\varphi_\kappa([\eta])$ and $x_2=\sigma_{x_1}\circ\hat{\pi}_{\kappa*}(a^\mu\partial_\mu)=S+x_1$, with $S^j\partial_j=\varphi_{\kappa*}(a^\mu\partial_\mu)$ as defined in (73).

Therefore, we get

$$\begin{aligned}&\bar{c}\circ\Phi^{-1}: M_4\times M_4\to R;\\ &\quad(x_1,x_2)\mapsto\bar{c}([(\varphi_\kappa^{-1}(x_1),(\sigma_{x_1}\circ\hat{\pi}_{\kappa*})^{-1}(x_2))])=\bar{c}([(\varphi_\kappa^{-1}(x_1),a^\mu\partial_\mu)]),\\ &\qquad=\varepsilon(a^\mu a_\mu),\end{aligned}$$

where $a^\mu\partial_\mu$ is given by $\hat{\pi}_{\kappa*}^{-1}(S^j\partial_j)$ with $S^j=x_2^j-x_1^j$. Using equation (78) we get

$$\varepsilon(a^\mu a_\mu)=\varepsilon\left(\frac{1}{K^2}S^jS_j\right)=\varepsilon(S^jS_j)=\varepsilon((x_2-x_1)^2)=c(x_1,x_2),$$

and thus

$$\bar{c}\circ\Phi^{-1}=c \quad \text{on} \quad \Phi(U)=M_4\times M_4\cong T(M_4). \tag{89}$$

Next, consider the mapping $\bar{t}$:

$$\begin{aligned}&\bar{t}\circ\Phi^{-1}: M_4\times M_4\to R,\\ &\quad(x_1,x_2)\mapsto\bar{t}([(\varphi_\kappa^{-1}(x_1),(\sigma_x\circ\hat{\pi}_{\kappa*})^{-1}(x_2)])=\bar{t}([(\varphi_\kappa^{-1}(x_1),a^\mu\partial_\mu)]),\\ &\qquad=\varepsilon(\eta^0a^5-\eta^5a^0)\end{aligned}$$

where $a^\mu\partial_\mu$ is given as above by $\hat{\pi}_{\kappa*}^{-1}(S^j\partial_j)$ with $S^j=x_2^j-x_1^j$.

Using equation (73) a simple calculation along the lines of Appendix A shows that

$$\varepsilon(\eta^0a^5-\eta^5a^0)=\varepsilon(S^0)=\varepsilon(x_2^0-x_1^0)=t(x_1,x_2).$$

Thus on the local chart (U, Φ) we have

$$\bar{t} \circ \Phi^{-1} = t. \tag{90}$$

Now let $[(q, a^\mu \partial_\mu)]$ be any point of $T(M_c^4)$. As the group $O^\uparrow(2,4)$ acts transitively on M_c^4, there exists a $w \in O^\uparrow(2,4)$ such that $w[q] \in U_\kappa$, and therefore, $[(q, a^\mu \partial_\mu)] \in U$. We define a local C_{15}-equivalent chart (V, Ψ) of $T(M_c^4)$ with $[(q, a^\mu \partial_\mu)] \in V$ in the following way:

$$V := w^{-1} U \quad \text{and} \quad \psi := \Phi \circ w,$$

i.e.

$$\psi : [(q, a^\mu \partial_\mu)] \mapsto (\varphi_k(w[q]), (\sigma \circ \pi_{k*})(w_* a^\mu \partial_\mu)).$$

Now let (z_1, z_2) be any element of $\Psi(V) = M_4 \times M_4$. Then we have

$$\bar{c} \circ \psi^{-1}(z_1, z_2) = \bar{c} \circ w^{-1} \circ \Phi^{-1}(z_1, z_2) = \bar{c}(w^{-1}(\Phi^{-1}(z_1, z_2)))$$
$$= \bar{c}(\Phi^{-1}(z_1, z_2)) = \bar{c} \circ \Phi^{-1}(z_1, z_2) = c(z_1, z_2),$$

because of equation (89) and the conformal invariance of $\bar{c}$.

For $\bar{t} \circ \Psi^{-1}$, finally, we get using equation (90) and part (a) of Theorem II:

$$\bar{t} \circ \Psi^{-1}(z_1, z_2) = \bar{t}(w^{-1}(\Phi^{-1}(z_1, z_2))) = \bar{t}(\Phi^{-1}(z_1, z_2)),$$
$$\bar{t} \circ \Phi^{-1}(z_1, z_2) = t(z_1, z_2), \quad \text{if} \quad \bar{c}(\Phi^{-1}(z_1, z_2)) = c(z_1, z_2) \geqslant 0.$$

Therefore everywhere on M_c^4 the functions $\bar{c}$ and $\bar{t}$ have locally all the properties of the functions c and t in Minkowski space.

The C_{15}-equivalent local charts play the same role as the Poincaré-equivalent inertial systems in M^4. Furthermore, it follows from the above construction that the "causality group", in the sense of Zeeman [33], is isomorphic to $(D_1 \otimes L_6) \otimes T_4$ on each chart which is equivalent to M_4.

In order to demonstrate the acausal structure of M_4^c *globally* we show that for any two points of M_c^4 each of them is in the forward time cone of the other.

DEFINITION.

$$L_{[q]}^+ = \{[(q, a^\mu \partial_\mu)] \in T_{[q]}(M_c^4) : \bar{c}([(q, a^\mu \partial_\mu)]) > 0,\ \bar{t}([(q, a^\mu \partial_\mu)]) > 0\},$$
$$L_{[q]}^- = \{[(q, a^\mu \partial_\mu)] \in T_{[q]}(M_c^4) : \bar{c}([(q, a^\mu \partial_\mu)]) > 0,\ \bar{t}([(q, a^\mu \partial_\mu)]) < 0\}.$$

$L_{[q]}^+$ is called the *forward time-cone at the point* $[q]$, $L_{[q]}^-$ is called the *backward time-cone at the point* $[q]$. From Theorem II it is clear that both $L_{[q]}^+$ and $L_{[q]}^-$ are invariant under conformal transformations in the following sense:

$$wL_{[q]}^+ = L_{w[q]}^+ \quad \text{and} \quad wL_{[q]}^- = L_{w[q]}^- \quad \text{for all} \quad w \in O^\uparrow(2,4).$$

Therefore we can introduce a time orientation on the manifold M_c^4 in the following way ([10], [28]).

DEFINITION. A piecewise differentiable curve γ: $\tau \in [0, 1] \mapsto \gamma(\tau) \in M_c^4$ is called *timelike* or *lightlike* if the tangent vectors $\gamma_*(\tau)$: $T_\tau(R) \to T_{\gamma(\tau)}(M_c^4)$ are timelike or lightlike respectively, that is $\bar{c}(\gamma(\tau), \gamma_*(\tau)(\partial/\partial\tau)) > 0$ or $= 0$. We say that a point $[p]$ is *earlier in time* than a point $[q]$ iff there exists a timelike curve γ, $\gamma(0) = [p]$, $\gamma(1) = [q]$, such that $(\gamma(\tau), \gamma_*(\tau)(\partial/\partial\tau)) \in L^+_{\gamma(\tau)}$ for all $\tau \in [0, 1]$; then we write $[p] < [q]$.

From our discussion of the functions $\bar{c}$ and $\bar{t}$ on $T(M_c^4)$ we see that the above definition is conformally invariant, i.e.

$$\textit{if } [p] < [q], \textit{ then } w[p] < w[q] \textit{ for all } w \in O^\uparrow(2, 4).$$

Let γ: $[0, 1]$ be a timelike curve with $\gamma(0) = [p]$ and $\gamma(1) = [q]$ and $\left(\gamma(\tau), \gamma_*(\tau)\left(\frac{\partial}{\partial\tau}\right)\right) \in L^+_{\gamma(\tau)}$ for all $\tau \in [0, 1]$.

Then the curve $\Gamma(\tau) := w \circ \gamma(\tau)$ is also timelike and, furthermore,

$$\left(w \circ \gamma(\tau), (w \circ \gamma)_*(\tau)\left(\frac{\partial}{\partial\tau}\right)\right) = \left(w\gamma(\tau), (w_* \circ \gamma_*)(\tau)\left(\frac{\partial}{\partial\tau}\right)\right) \in wL^+_{\gamma(\tau)} = L^+_{w \circ \gamma(\tau)} = L^+_{\Gamma(\tau)}.$$

Therefore, the manifold M_c^4 admits a conformally invariant time orientation. On the other hand, we know ([10], [28]) that every compact Lorentz manifold equipped with a time orientation has at least one closed timelike curve, i.e. there exists a point $[p]$ such that $[p] < [p]$. Thus for every point $[q] \in M_c^4$ there exists a closed timelike curve through $[q]$, because the group $O^\uparrow(2, 4)$ acts transitively on M_c^4. We can even show that any two points $[q]$ and $[p]$ of M_c^4 can be connected by a timelike curve γ, $\gamma(0) = [q]$, $\gamma(1) = [p]$, such that the tangent vectors at any point of the curve lie in the forward time cone, i.e. for any two points $[p]$ and $[q] \in M_c^4$ it follows that

$$[p] < [q].$$

If we denote by the future $F_{[p]}$ of $[p]$ the set of all points $[q]$ with the property $[q] > [p]$, then we have $F_{[p]} = M_c^4$. Thus the manifold M_c^4 is highly acausal *globally*. The proof goes as follows: First, consider the case where $[p]$, $[q] \in U_\kappa$: let $x_1 = \varphi_\kappa([p])$ and $x_2 = \varphi_\kappa([q])$ be the local coordinates of the two points $[p]$ and $[q]$. Without loss of generality we can assume that $x_1 \equiv 0$ in M^4. Now there are the three cases:

a) $x_2^2 = 0$, b) $x_2^2 < 0$, c) $x_2^2 > 0$.

If $x_2^2 > 0$, then we know that there exists a timelike curve $\bar{\gamma}$: $[0, 1] \to M^4$, $\bar{\gamma}(0) = 0$; $\bar{\gamma}(1) = x_2$ such that the tangent vector at each point $\bar{\gamma}(\tau)$ is in the forward light cone V_+ in Minkowski space. Consider the curve Γ: $[0, 1] \to M_c^4$ defined by $\Gamma(\tau) := (\varphi_\kappa^{-1} \circ \bar{\gamma})(\tau)$. This curve has the property $\Gamma(0) = [p]$, $\Gamma(1) = [q]$ and, because of the Theorem II, it follows immediately that at any point $\Gamma(\tau)$ the tangent vector $\Gamma_*(\tau)(\partial/\partial\tau)$ is in the forward time cone $L^+_{\Gamma(\tau)}$ for all $\tau \in [0, 1]$.

If $x_2^2 < 0$, then there exists a special conformal transformation

$$w(c) \in O^\uparrow(2, 4)/Z_2 : \; x_2^i \mapsto x_2'^i = \frac{x_2^i - c^i x_2^2}{\sigma(x_2, c)}, \quad \text{with} \quad x_2'^2 > 0.$$

Therefore there exists a curve $\bar{\gamma}(\tau)$: $\bar{\gamma}(0)=0$; $\bar{\gamma}(1)=x_2'$ with $\bar{\gamma}_*(\tau)(\partial/\partial\tau)\in V_+$, the forward time cone in Minkowski space, $\tau\in[0,1]$. Consider now the curve Γ: $[0,1]\to M_c^+$, defined by $\Gamma(\tau):=w(c)^{-1}\circ\varphi_\kappa^{-1}\circ\bar{\gamma}(\tau)$. Then it follows that $\Gamma(0)=[p]$ and $\Gamma(1)=[q]$.

Theorem II gives again $\Gamma_*(\tau)(\partial/\partial\tau)\in L^+_{\Gamma(\tau)}$ for all $\tau\in[0,1]$. If $x_2^2=0$, then it follows for the coordinates $[\eta_1^\mu]$ and $[\eta_2^\mu]$ of the points $[p]$ and $[q]$, respectively, that $\eta_1,\eta_2=0$. We shall treat this case at the end.

Next let us take any two points $[p]$, $[q]\in M_c^4$ such that $\eta_1^\mu\cdot\eta_{2\mu}\neq 0$. There exists a conformal transformation $w\in O^\uparrow(2,4)$ with $w([p])$, $w([q])\in U_\kappa$. As we have already shown, there exists a curve $\tilde{\Gamma}$: $[0,1]\to M_c^4$, $\tilde{\Gamma}(0)=w([p])$, $\tilde{\Gamma}(1)=w([q])$ such that

$$\tilde{\Gamma}_*(\tau)\left(\frac{\partial}{\partial\tau}\right)\in L^+_{\tilde{\Gamma}(\tau)}\qquad\text{for all }\tau\in[0,1].$$

Then define a new curve Γ: $[0,1]\to M_c^4$ by $\Gamma(\tau)=w^{-1}\circ\tilde{\Gamma}(\tau)$, with $\Gamma(0)=[p]$, $\Gamma(1)=[q]$. The conformal invariance of the functions $\bar{c}$, $\bar{t}$ again shows that $\Gamma_*(\tau)(\partial/\partial\tau)\in L^+_{\Gamma(\tau)}$ for all $\tau\in[0,1]$. We have shown therefore that any two points $[p]$, $[q]\in M_c^4$ with $\eta_1^\mu\,\eta_{2\mu}\neq 0$ can be connected by a timelike curve with tangents in the forward time cone. If the points $[p]$ and $[q]\in M_c^4$ are such that $\eta_1^\mu\,\eta_{2\mu}=0$, then choose a point $[r]\in M_c^4$, with $r^\mu\eta_{1\mu}\neq 0$, $r^\mu\eta_{2\mu}\neq 0$. Then we can connect the point $[p]$ with the point $[r]$ and the point $[r]$ with the point $[q]$ by timelike curves with tangent vectors always in the forward time cone.

Acknowledgements

We are indebted to Dr. O. Loos for some mathematical advice at the beginning of this work. We also thank Professor R. Haag and Dr. Roberts for critical comments on the first version of this paper.

Two of us (T.H.G. and D.H.M.) thank the Bundesministerium für Bildung und Wissenschaft for financial support. Finally, one of us (T.H.G.) is indebted for a fellowship of the Deutsche Akademische Austauschdienst.

Appendix A

In this Appendix we prove the theorem which was used in connection with our introduction of a causal structure in Sections 7 and 8.

Let η_1^μ and η_2^μ be the coordinates of two points in $\boldsymbol{R}_0^6$; then we write

$$[\mu,\nu]\equiv\eta_1^\mu\eta_2^\nu-\eta_2^\mu\eta_1^\nu,\qquad \nu,\mu\in\{0,1,2,3,4,5\},$$

$$z^k=[k,5],\qquad k=0,1,2,3.$$

LEMMA. *Let* $\eta_i\in Q_0^5$, $k(\eta_1,\eta_2)\equiv\eta_1\cdot\eta_2=0$. *If* $[0,5]=[0,4]$, *then we have* $\eta_1=\alpha\eta_2$, $\alpha\in\boldsymbol{R}_0$, *or* $[4,5]=0$.

Proof: From $\eta_1 \cdot \eta_2 = 0$ and $\eta_i \in Q_0^5$ we get

$$[0,5]^2 = \sum_{k=1}^{4} [k,0]^2 = \sum_{k=1}^{4} [k,5]^2. \tag{A1}$$

Suppose $[0,5]=0$. Then (A1) implies $[0,4]=[0,k]=[k,5]=0$. As $\eta_i \in Q_0^5$, we may assume without loss of generality that $\eta_1^0 \neq 0$. In consequence, we have $\eta_2^0 \neq 0$ and $\eta_2 = \frac{\eta_1^0}{\eta_1^0} \cdot \eta_1$. Let $[0,5] \neq 0$. Then we may assume once more that $\eta_1^0 \neq 0$.

Equation (A1) gives us $\eta_2^k = \eta_2^0\, \eta_1^k / \eta_1^0$, $k=1,2,3$, and $\eta_2^4 = ([0,5]+\eta_2^0 \eta_1^4)/\eta_1^0$. As $\eta_1 \cdot \eta_2 = 0$, it follows that $\eta_1^4 = \eta_1^5$, and therefore $[4,5]=0$.

We now prove

THEOREM I. *Let* $\eta_1 \cdot \eta_2 = 0$. *Then* $\operatorname{sgn}(\eta_1^0\, \eta_2^5 - \eta_2^0\, \eta_1^5) \equiv \varepsilon(0\ 5)$ *is invariant under the action of the group* $O^\uparrow(2,4)$ *iff* $\eta_i \cdot \eta_i \geqslant 0$, $i=1,2$.

Proof: First, suppose that $\eta_i \cdot \eta_i \geqslant 0$. The condition $\eta_1 \cdot \eta_2 = 0$ implies (A1) for $\eta_i \cdot \eta_i = 0$, $i=1,2$, or the inequalities

$$[0,5]^2 > \sum_{k=1}^{4} [k,0]^2, \quad [0,5]^2 > \sum_{k=1}^{4} [k,5]^2, \tag{A2}$$

if at least one of the points η_i satisfies $\eta_i \cdot \eta_i > 0$, whereas $\eta_j \cdot \eta_j \geqslant 0$, $j \neq i$.

The relations (A1) and (A2) remain valid under the action of the group $O^\uparrow(2,4)$, because the scalar product $\eta_1 \cdot \eta_2$ is invariant under $O^\uparrow(2,4)$.

We now turn to the problem of the invariance of $\varepsilon(0\ 5)$ under the action of $O^\uparrow(2,4)$. Its invariance under the subgroup $L_6^\uparrow[\Lambda]$ can be inferred from $\varepsilon(z^0) = \varepsilon((\Lambda z)^0)$, $\Lambda \in L_6^\uparrow[\Lambda]$, which holds because $z^k z_k \geqslant [5,4]^2 \geqslant 0$.

The invariance of $\varepsilon(0\ 5)$ under the dilatations can easily be verified.

In the case of the translations T_a, $a=(a^0, \ldots, a^3)$, we define $T_{a^k}[0,5] \equiv [0,5]_k'$ and get

$$\begin{aligned} [0,5]_0' &= \left(1 + \frac{(a^0)^2}{2}\right)[0,5] + \frac{(a^0)^2}{2}[4,0] + a^0[4,5], \\ [0,5]_i' &= \left(1 + \frac{(a^i)^2}{2}\right)[0,5] + \frac{(a^i)^2}{2}[4,0] + a^i[0,i], \qquad i=1,2,3, \end{aligned} \tag{A3}$$

as well as

$$[0,5]-[0,4] = [0,5]' - [0,4]'. \tag{A4}$$

If $\eta_i \cdot \eta_i > 0$, $i=1$ or 2, equation (A2) implies the invariance of $\varepsilon(0\ 5)$ under the translations $T_a \in O^\uparrow(2,4)$.

If $\eta_i \cdot \eta_i = 0$, $i=1,2$ we have either $[0,5]=[0,4]$ or $[0,5] \neq [0,4]$. In the first of these cases our lemma implies $\eta_1 = \alpha \eta_2$ or $[4,5]=0$ and $\eta_2^k = \eta_1^k\, \eta_2^0 / \eta_1^0$, $k=1,2,3$. If $\eta_1 = \alpha \eta_2$, the invariance of $\varepsilon(0\ 5)$ is trivial. As for the second possibility, the invariance of $\varepsilon(0\ 5)$ follows from (A3).

In the case of $[0, 5] \neq [0, 4]$, equation (A1) gives $[0, 5] = -[0, 4]$ or $|[0, 5]| > |[0, 4]|$. The invariance of equation (A1) under the group $O^{\uparrow}(2, 4)$ and the relation (A4) imply $[0, 5]' = -[0, 4]'$ or $|[0, 5|' > |[0, 4]'|$. From these properties the translation invariance of $\varepsilon(0\ 5)$ follows immediately.

As the inversion I_r leaves $\varepsilon(0\ 5)$ invariant, too, and since the special conformal transformations are given in the form of compositions $I_r\ T_c\ I_r$, $\varepsilon(0\ 5)$ is invariant under the group $SC_4[c]$. This completes the proof of the theorem in one direction.

Second, let us assume that $\varepsilon(0\ 5)$ is invariant under the group $O^{\uparrow}(2, 4)$. Suppose $\eta_1 \cdot \eta_1 < 0$. But then we can always find an element of $O^{\uparrow}(2, 4)$ which transforms the coordinates into such that $\eta_1^{0'} = 0$, $\eta_1^{5'} = 0$. This is already enough to demonstrate that $\varepsilon(0\ 5)$ cannot be an invariant if $\eta_i \cdot \eta_i < 0$.

The validity of the next theorem can be also verified easily.

THEOREM Ia. *Let* $\eta_i \cdot \eta_i \geqslant 0$, $\eta_i \in R_0^6$. *Then* $\varepsilon(0\ 5)$ *is invariant under the action of the group* $O^{\uparrow}(2, 4)$, *iff* $\eta_1 \cdot \eta_2 = 0$.

Proof: According to our discussion above it suffices to show that $\eta_1 \cdot \eta_2 = 0$ is a necessary condition. Assume $\eta_1 \cdot \eta_2 \neq 0$. It then follows that $z^k \cdot z_k \neq [4, 5]^2 \geqslant 0$, i.e. z^k may be a spacelike vector and then $\varepsilon(0\ 5)$ is in general not invariant under the group $L_6^{\uparrow}\ [\Lambda]$, contrary to our assumption about the $O^{\uparrow}(2, 4)$ invariance of $\varepsilon(0\ 5)$. It can be seen from the following example that such spacelike vectors exist:

$$\eta_1 = (0, \eta_1^1 \neq 0, 0, 0, 0, \eta_1^5 \neq 0), \quad \eta_1 \cdot \eta_1 \geqslant 0;$$

$$\eta_2 = (0, 0, 0, 0, \eta_2^4 \neq 0, \eta_2^5 \neq 0), \quad \eta_2 \cdot \eta_2 \geqslant 0.$$

REFERENCES

[1] Brandt, R., G. Preparata, Nucl. Phys. **B27** (1971), 541.
[2] Brower, R., H. Weyl, Amer. J. Math. **57** (1935), 425.
[3] Castell, J., Nucl. Phys. **B13** (1969), 231.
[4] Eisenhart, L. P., *Riemannian geometry*, Princeton University Press, Princeton, 1949.
[5] Ellis, J., Review paper presented at the Coral Gables Conf. on High Energy, Jan. 20–22, 1971.
[6] Flato, M., J. Simon, D. Sternheimer, Ann. Phys. **61** (1970), 78.
[7] Frishman, Y., Phys. Rev. Lett. **25** (1970), 966.
[8] —, Ann. Phys. (N.Y.) **66** (1971), 373.
[9] Gamba, A., G. Luzzato, Nuovo Cim. **33** (1964), 1732.
[10] Geroch, R., Proc. Intern. School E. Fermi, Course **XLVII**, 1971, 71.
[11] Helgason, S., *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.
[12] Hirzebruch, F., *Topological methods in algebraic geometry*, Third edition, Springer Verlag, Berlin–Heidelberg–New York, 1966.
[13] Husemoller, D., *Fibre bundles*, MacGraw-Hill Book Company, New York, 1966.
[14] *Informal Frascati Meeting on Conformal Symmetry*, ed. R. Gatto, Wiley–Interscience, 1973.
[15] Kastrup, H. A., Physics Letters 3 (1962), 78.
[16] —, Nuclear Physics **58** (1964), 561.
[17] —, Annalen der Phys. **9** (1962), 388.
[18] —, Phys. Rev. **150** (1966), 1183.

[19] Kobayashi, S., K. Nomizu, *Foundations of differential geometry*, Vol. 1, Interscience Publ., New York, 1963.
[20] Kuiper, N. H., Ann. Math. **50** (1949), 916.
[21] Mack, G., Nucl. Phys. **B35** (1971), 592.
[22] Mack, G., I. Todorov, *Conformal invariant Green function without ultraviolet divergences*, Phys. Rev D **8** (1973), 1764.
[23] Migdal, A. A., Physics Lett. **37B** (1971), 98.
[24] —, *ibid.* **37B** (1971), 386.
[25] *Proceedings of the symposium on basic questions in elementary particle physics*, June 1971, Max-Planck-Institut für Physik und Astrophysik, München.
[26] Reeh, H., *Symmetries, currents and infinitesimal generators*, Preprint of the Max-Planck-Institute for Physics, Munich, 1971.
[27] Trautman, A., Reports Math. Phys. **1** (1970), 1.
[28] Uhlenbeck, K., *Lorentz geometry*, College on Global Analysis and its Applications, Trieste, ICTP, July–August, 1972.
[29] Wess, J., Nuovo Cim. **18** (1960), 1086.
[30] Weyl, H., *Mathematische Analyze des Raumproblems*, Lectures given at Barcelona and Madrid, Verlag von Julius Springer, Berlin, 1923.
[31] Wilson, K., Phys. Rev. **179** (1969), 1499.
[32] Wolf, J., *Spaces of constant curvature*, McGraw-Hill Book Company, New York, 1967.
[33] Zeeman, E C., J. Math. Phys. **5** (1964), 490.
[34] Zimmermann, W., in *Quanten und Felder*, ed. H. P. Dürr, Vieweg Sohn, Braunschweig, 1971