HAMILTON–JACOBI THEORIES FOR STRINGS

H.A. KASTRUP and M. RINKE
Institut für Theoretische Physik, RWTH Aachen, 51 Aachen, Fed. Rep. Germany

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Carathéodory's Hamilton–Jacobi theory for fields is applied to the relativistic string. It can be used to establish relations between string motions and Maxwell fields of rank two.

To establish relations between relativistic string models [1] and gauge theories is considered to be an important problem in strong interaction physics [2]. Let

$$x^\mu = x^\mu(\tau^1, \tau^2) = x^\mu(\tau)\,, \quad \mu = 0, 1, 2, 3\,;$$

$$-\infty < \tau^1 < \infty\,, \quad 0 \leqslant \tau^2 \leqslant \pi\,,$$

be the two-dimensional surface $\Sigma^{(2)}$ in Minkowski space M^4 "swept out" by the string. The action A_I for the string motion, which is invariant under arbitrary reparametrizations

$$\tau^j \to \hat{\tau}^j(\tau^1, \tau^2)\,, \quad j = 1, 2\,,$$

$$\partial(\hat{\tau}^1, \hat{\tau}^2)/\partial(\tau^1, \tau^2) \neq 0\,,$$

of the surface $\Sigma^{(2)}$, can be written – with an appropriate choice of the units – as

$$A_I = \int d\tau^1\, d\tau^2\, L_I(v)\,,$$

$$L_I(v) = (-\tfrac{1}{2} v_{\mu\nu} v^{\mu\nu})^{1/2} = [(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2]^{1/2}\,, \qquad (1)$$

where

$$v^{\mu\nu} = \dot{x}^\mu x'^\nu - \dot{x}^\nu x'^\mu\,,$$

$$\dot{x}^\mu := \partial_{(1)} x^\mu\,, \quad x'^\mu := \partial_{(2)} x^\mu\,,$$

$$\partial_{(j)} := \partial/\partial\tau^j\,, \quad j = 1, 2\,,$$

$$a\cdot b = g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - \boldsymbol{a}\cdot\boldsymbol{b}\,, \quad a^2 := a\cdot a\,.$$

There have been several attempts [3] to relate Plücker's coordinates $v^{\mu\nu}$ of the surface $\Sigma^{(2)}$ to gauge fields $F^{\mu\nu}(x)$. One of us [4] suggested the relation

$$F^{\mu\nu}(x(\tau)) = \lambda v^{\mu\nu}(\tau)\,, \quad \lambda = \text{const.}\,, \qquad (2)$$

on $\Sigma^{(2)}$, where $(F^{\mu\nu})$ is a Maxwell field of rank two, i.e. $F^{\mu\nu}$ has to satisfy [5]

$$^*F_{\mu\nu} F^{\mu\nu} = 4\boldsymbol{E}\cdot\boldsymbol{B} = 0\,, \quad ^*F_{\mu\nu} := \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}\,. \qquad (3)$$

For the special string motion

$$x^0 = \tau^1, \quad x^1 = A(\tau^2 - \tfrac{1}{2}\pi)\cos\omega\tau^1\,,$$

$$x^2 = A(\tau^2 - \tfrac{1}{2}\pi)\sin\omega\tau^1\,, \quad x^3 = 0\,, \quad A\omega = 2/\pi\,, \qquad (4)$$

relation (2) leads to the electromagnetic fields [4]

$$\boldsymbol{E} = -\lambda A(\cos\omega t, \sin\omega t, 0)\,, \quad \boldsymbol{B} = \lambda A(0, 0, \omega\rho)\,,$$

$$t = x^0\,, \quad \rho := [(x^1)^2 + (x^2)^2]^{1/2}\,,$$

$$\text{curl}\,\boldsymbol{E} = 0\,, \quad \text{div}\,\boldsymbol{E} = 0\,,$$

$$\text{div}\,\boldsymbol{B} = 0\,, \quad \text{curl}\,\boldsymbol{B} = \omega\lambda A(x^2/\rho, -x^1/\rho, 0)\,. \qquad (5a)$$

or [6]

$$\boldsymbol{E} = -\lambda A(x^1/\rho, x^2/\rho, 0)\,, \quad \boldsymbol{B} = \lambda A(0, 0, \omega\rho)\,,$$

$$\text{curl}\,\boldsymbol{E} = 0\,, \quad \text{div}\,\boldsymbol{E} = -\lambda A/\rho\,,$$

$$\text{div}\,\boldsymbol{B} = 0\,, \quad \text{curl}\,\boldsymbol{B} = \omega\lambda A(x^2/\rho, -x^1/\rho, 0)\,. \qquad (5b)$$

The ambiguity of the correspondence between string motions and electromagnetic fields, for which the eqs. (5a) and (5b) provide an example, is due to the fact that the equality (2) is required to hold on the surface $\Sigma^{(2)}$ only and that there are many possibilities to extend the fields (2) from $\Sigma^{(2)}$ to the whole Minkowski space M^4.

There is, however, a "canonical" extension [6] which uses Carathéodory's Hamilton–Jacobi theory for fields [7] ‡1: Suppose, we have a parametrization of the surface $\Sigma^{(2)}$ such that $L_I = 1$ on $\Sigma^{(2)}$. The functions $S^j(x)$, $j = 1, 2$, on M^4 define two-dimensional "wave fronts" $S^j(x) = \text{const.}$, $j = 1, 2$, "transversal" to $\Sigma^{(2)}$, if

$$-v_{\mu\nu}/L_I = \partial_\mu S^1 \partial_\nu S^2 - \partial_\nu S^1 \partial_\mu S^2 =: S_{\mu\nu}(x) \quad \text{on } \Sigma^{(2)} ,$$

$$\partial_\mu := \partial/\partial x^\mu , \qquad (6)$$

and if the functions $S^j(x)$ obey the Hamilton–Jacobi (H–J) equation

$$(-\tfrac{1}{2} S_{\mu\nu} S^{\mu\nu})^{1/2} = 1 . \qquad (7)$$

The relations (2) and (6) with $L_I = 1$ suggest to define [6]

$$F_{\mu\nu}(x) = -\lambda S_{\mu\nu}(x) \qquad (8)$$

on M^4. The extension (8) fulfills the homogeneous Maxwell equations $\partial_\mu {}^*F^{\mu\nu} = 0$, but one will have $\partial_\mu F^{\mu\nu} \neq 0$ in general. Functions $S^j(x)$, $j = 1, 2$, which obey the H–J equation (7) and which fulfill the transversality conditions (6) for the motion (4), are

$$S^1 = (1 - \omega^2\rho^2)^{-1/2}[x^0 - \omega\rho^2 \operatorname{arctg}(x^2/x^1)] ,$$

$$S^2 = \rho . \qquad (9)$$

The fields (8) calculated from the functions $S^j(x)$ in (9) are those of eqs. (5b) divided by $A(1 - \omega^2\rho^2)^{1/2}$.

Recently, Nambu proposed a Hamilton–Jacobi formalism [9] for strings which starts from the action [10]

$$A_S = \int d\tau^1 \, d\tau^2 \, L_S , \quad L_S = \tfrac{1}{4} v_{\mu\nu} v^{\mu\nu} . \qquad (10)$$

If $L_I > 0$, this action leads to the same field equations as the action (1) [10], however, the action (10) is no longer invariant under arbitrary reparametrizations, but only under those for which $\partial(\hat\tau^1, \hat\tau^2)/\partial(\tau^1, \tau^2) = 1$.

As an analogue to the canonical Hamilton–Jacobi Cartan form $dS = -H\, dt + p_j\, dq^j$ in mechanics, Nambu suggests for strings the two-form

$$dS^1(x,\tau) \wedge dT^1(x,\tau) + dS^2(x,\tau) \wedge dT^2(x,\tau)$$
$$= \tfrac{1}{2} \hat p_{\mu\nu} dx^\mu \wedge dx^\nu - H\, d\tau^1 \wedge d\tau^2 , \qquad (11)$$

‡1 A detailed account will be contained in a forthcoming review [8].

where

$$\hat p_{\mu\nu} = v_{\mu\nu} , \quad H = \tfrac{1}{4} \hat p_{\mu\nu} \hat p^{\mu\nu} = L_S .$$

In the following we want to point out that the H–J relation (11) in general has integrability problems which have to do with the rank of the two-form (11) and we shall show that Carathéodory's H–J theory for fields leads to an appropriate theory for the action (10), too.

As the notion of the rank r of a differential form ω is important here – but not so well-known – let us recall its essential elements [11]: Locally a p-form ω^p on an n-dimensional manifold $M^{(n)}$ with local coordinates y^ν, $\nu = 1, ..., n$, $n \geqslant p$, is a linear combination of p-fold exterior products $dy^{\nu_1} \wedge ... \wedge dy^{\nu_p}$ of the differentials dy^ν. The rank of ω^p is the minimal number of linearly independent one-forms $\Theta^{(\rho)} = f_\nu^{(\rho)}(y)\, dy^\nu$, $\rho = 1, ..., r$, by which ω^p can be expressed. Obviously $r \geqslant p$. The minimal rank $r = p$ is obtained if $\omega^p = \Theta^{(1)} \wedge ... \wedge \Theta^{(p)}$. In that case ω^p is called "simple" or "decomposable". At each point (y) the r one-forms $\Theta^{(\rho)} =$ span an r-dimensional subspace $S_y^{(r)*}$ of the n-dimensional space $T_y^{(n)*}$ of cotangent vectors. These r-dimensional spaces determine an $(n-r)$-dimensional "differential" system $S^{(n-r)}$ of (dual) vector fields $Y = Y^\nu(y)\, \partial_\nu$, defined by

$$\Theta^{(\rho)}(Y) = f_\nu^{(\rho)} Y^\nu = 0 , \quad \rho = 1, ..., r .$$

Let $Y^{(\sigma)}$, $\sigma = 1, ..., n-r$, be a basis of these vector fields "associated" with the form ω^p. If ω^p is closed, i.e. if $d\omega^p = 0$ (which is the case of interest for us; the more general case with $d\omega^p \neq 0$ is discussed in ref. [11]), then the system $S^{(n-r)} = \{Y^{(\sigma)}\}$ is completely integrable, i.e. there is an $(n-r)$-dimensional submanifold $N^{(n-r)} \subset M^{(n)}$ the tangent spaces of which at each point (y) are spanned by $S_y^{(n-r)}$. In addition the one-forms $\Theta^{(\rho)}$ can be chosen to be total differentials $\Theta^{(\rho)} = df^{(\rho)}(y)$ and the $(n-r)$-dimensional integral manifolds $N^{(n-r)}$ are defined by $f^{(\rho)}(y) = \text{const.}$, $\rho = 1, ..., r$.

In our context p is the number of independent variables, $n - p$ the number of field variables and $n - r$ the dimension of the "associated" H–J "wave fronts".

In mechanics the canonical one-form $\Theta = -H\, dt + p_j\, dq^j$ of a system with f degrees of freedom obviously has rank one. Its H–J theory is characterized by the property $\Theta = dS(t, q)$ and the f-dimensional integral manifolds "associated" with the form Θ are the "wave fronts" $S(t, q) = \text{const.}$ That is, the dimension of the wave fronts in mechanics is the same as the number of dependent variables q^j.

For a field theory with two independent variables and n dependent (field) variables a corresponding H–J theory with n-dimensional wave fronts requires the basic canonical two-form to have rank two!

As the form (11) has rank four, the wave fronts defined by $S^j(x,\tau) = \text{const.}$, $T^j(x,\tau) = \text{const.}$, $j = 1, 2$, are in general not four but $(6-4=2)$-dimensional! It does not help to put two of these functions equal to zero, because the rhs of eq. (11) in general has rank four.

The only H–J theory for our system which has rank two is that of Carathéodory! As it is based on one of several possibilities for defining a canonical framework in our example, we first recall the general case, due to the Belgian mathematician Lepage [12], and shall then discuss the special theory of Carathéodory:

Let us denote by v_j^μ the quantity which is equal to $\partial_{(j)} x^\mu$ on the surface $\Sigma^{(2)}$, i.e. the one-forms $\omega^\mu = dx^\mu - v_j^\mu \, d\tau^j$ vanish on $\Sigma^{(2)}$. This implies that the lagrangian two-form

$$\omega = L \, d\tau^1 \wedge d\tau^2 \tag{12}$$

is only one representative – as far as the extremals are concerned – of an equivalence class of two-forms, the most general one of which can be written as

$$\Omega = \omega + h_\mu^1 \omega^\mu \wedge d\tau^2 + h_\mu^2 \, d\tau^1 \wedge \omega^\mu + \tfrac{1}{2} h_{\mu\nu} \omega^\mu \wedge \omega^\nu ,$$

$$h_{\mu\nu} = -h_{\nu\mu} , \tag{13}$$

where the coefficients h_μ^j, $h_{\mu\nu}$ in general can be functions of τ^j, x^μ and v_j^μ. The one-forms ω^μ generate an ideal I in the algebra of differential forms on the six-dimensional space of variables τ^j, x^μ.

The existence of a H–J theory requires $d\Omega = 0 \pmod{I}$. Since

$$d\Omega = (\partial L/\partial v_j^\mu - h_\mu^j) \, dv_j^\mu \wedge d\tau^1 \wedge d\tau^2 + 0 \pmod{I} ,$$

$d\Omega \equiv 0 \pmod{I}$ is equivalent to

$$h_\mu^j = \partial L/\partial v_j^\mu =: \Pi_\mu^j . \tag{14}$$

In mechanics the (canonical) one-form corresponding to the two-form (13) combined with the property (14) is

$$\Omega = L \, dt + p_a \omega^a = L \, dt + p_a (dq^a - v^a \, dt)$$

$$= -(v^a p_a - L) \, dt + p_a \, dq^a = -H \, dt + p_a \, dq^a . \tag{15}$$

Here a term corresponding to $h_{\mu\nu}$ is not possible! Eqs. (15) show that the Legendre transformation $v^a \to p_a$, $L \to H$ can be implemented by a change of basis $\omega^a \to dq^a$ in the space of one-forms spanned by dt and ω^a or dq^a. By generalizing this procedure to the two-form (13) we replace ω^μ by $dx^\mu - v_j^\mu \, d\tau^j$ and identify the Hamilton function H with the resulting negative coefficient of $d\tau^1 \wedge d\tau^2$ and the canonical momentum p_μ^j with the coefficient of $\epsilon_{jk} \, dx^\mu \wedge d\tau^k$, $\epsilon_{jk} = -\epsilon_{kj}$, $\epsilon_{12} = 1$. The result is

$$\Omega = -H \, d\tau^1 \wedge d\tau^2 + p_\mu^j \epsilon_{jk} \, dx^\mu \wedge d\tau^k + \tfrac{1}{2} h_{\mu\nu} dx^\mu \wedge dx^\nu ,$$

$$H = \Pi_\mu^j v_j^\mu - \tfrac{1}{2} h_{\mu\nu} (v_1^\mu v_2^\nu - v_2^\mu v_1^\nu) - L ,$$

$$p_\mu^1 = \Pi_\mu^1 - h_{\mu\nu} v_2^\nu , \quad p_\mu^2 = \Pi_\mu^2 + h_{\mu\nu} v_1^\nu . \tag{16}$$

As the functions $h_{\mu\nu}$ in general are arbitrary, one sees that there is a wide range of possibilities for defining canonical momenta. The different canonical theories may be classified according to rank r, $2 \leqslant r \leqslant 6$, of the two-form (13) associated with a given choice of the functions $h_{\mu\nu}$. The conventional choice in physics is $h_{\mu\nu} = 0$, implying $r = 4$ for the form (13) which in this case can be expressed by the four one-forms $d\tau^j$, $L \, d\tau^j + \Pi_\mu^j \omega^\mu$, $j = 1, 2$. Thus the conventional canonical framework in general will not lead to four-dimensional wave fronts! The minimal rank $r = 2$ is only realized in Carathéodory's canonical theory, which is defined as follows:

With

$$a^j = L \, d\tau^j + \Pi_\mu^j \omega^\mu = T^j{}_k \, d\tau^k + \Pi_\mu^j \, dx^\mu ,$$

$$T^j{}_k = \Pi_\mu^j v_k^\mu - \delta_k^j L ,$$

and

$$\Theta^j = -H \, d\tau^j + p_\mu^j \, dx^\mu ,$$

the two-form (13) in Carathéodory's case is given by

$$\Omega_c = L^{-1} a^1 \wedge a^2 = -H^{-1} \Theta^1 \wedge \Theta^2 . \tag{17}$$

It obviously has rank two and from the second equality (17) we obtain

$$H = -L^{-1} |T^j{}_k| , \quad |A^j{}_k| := \det(A^j{}_k) , \tag{18a}$$

$$p_\mu^j = -L^{-1} \bar{T}^j{}_k \Pi_\mu^k \quad \text{or} \quad H \Pi_\mu^j = T^j{}_k p_\mu^k , \tag{18b}$$

where $\bar{T}^j{}_k$ is the algebraic complement of $T^j{}_k$, i.e. $\bar{T}^j{}_k T^k{}_l = \delta_l^j |T^i{}_k|$. Furthermore:

$$h_{\mu\nu} = L^{-1} (\Pi_\mu^1 \Pi_\nu^2 - \Pi_\mu^2 \Pi_\nu^1) = -H^{-1} (p_\mu^1 p_\nu^2 - p_\mu^2 p_\nu^1) . \tag{18c}$$

From the lagrangian

$$L_S = \tfrac{1}{4} v_{\mu\nu} v^{\mu\nu} = \tfrac{1}{2}[(v_1)^2(v_2)^2 - (v_1 \cdot v_2)^2] ,$$

we get

$$\Pi^1_\mu = g_{\mu\nu}[v_1^\nu (v_2)^2 - v_2^\nu (v_1 \cdot v_2)] ,$$
$$\Pi^2_\mu = g_{\mu\nu}[v_2^\nu (v_1)^2 - v_1^\nu (v_1 \cdot v_2)] , \qquad (19)$$

which give $T^1{}_1 = L_S, T^1{}_2 = 0, T^2{}_1 = 0, T^2{}_2 = L_S$ and therefore

$$H = -L_S , \quad p^j_\mu = -\Pi^j_\mu . \qquad (20\text{a,b})$$

Because

$$\Pi^1_\mu \Pi^2_\nu - \Pi^2_\mu \Pi^1_\nu = 2 v_{\mu\nu} L_S , \qquad (21)$$

we have, see eqs. (20b),

$$p_{\mu\nu} p^{\mu\nu} = 4 v_{\mu\nu} v^{\mu\nu} L_S^2 = 16 L_S^3 , \quad p_{\mu\nu} := p^1_\mu p^2_\nu - p^2_\mu p^1_\nu .$$

Combining this with eq. (20a) we obtain

$$H = -(\tfrac{1}{16} p_{\mu\nu} p^{\mu\nu})^{1/3} . \qquad (22)$$

The H–J equation we get from

$$dS^1 \wedge dS^2 = -H^{-1} \Theta^1 \wedge \Theta^2 , \quad S^j = S^j(x, \tau) , \qquad (23)$$

which implies

$$|\partial_{(j)} S^k| + H(x, p) = 0 , \qquad (24\text{a})$$
$$p^1_\mu = \psi^1_\mu(x, \tau) := \partial_{(2)} S^2 \partial_\mu S^1 - \partial_{(2)} S^1 \partial_\mu S^2 ,$$
$$p^2_\mu = \psi^2_\mu(x, \tau) := \partial_{(1)} S^1 \partial_\mu S^2 - \partial_{(1)} S^2 \partial_\mu S^1 ,$$
$$p_{\mu\nu} = \psi_{\mu\nu}(x, \tau) := |\partial_{(j)} S^k| (\partial_\mu S^1 \partial_\nu S^2 - \partial_\mu S^2 \partial_\nu S^1) . \qquad (24\text{b})$$

As an application we discuss the "wave front" functions $S^j(x, \tau)$ associated with the "rigid" motion

$$x^0 = \tau^1 , \quad x^1 = \rho(\tau^2) \cos \omega\tau^1 ,$$
$$x^2 = \rho(\tau^2) \sin \omega\tau^1 , \quad x^3 = 0 ,$$
$$d\rho/d\tau^2 = (1 - \omega^2\rho^2)^{-1/2} , \quad 0 \leqslant \rho \leqslant \omega^{-1} , \qquad (25)$$

which is obtained from eqs. (4) by a reparametrization [6]

$$\hat\tau^1 = \tau^1 , \quad \hat\tau^2 = \int_0^{\tau^2} L_I(v(\tau^1, \bar\tau^2)) \, d\bar\tau^2 ,$$

$$L_I = (1 - \omega^2\rho^2)^{1/2} , \quad A = 1 ,$$

and denoting $\hat\tau^j$ again by τ^j. From eqs. (25) we obtain

$$(\dot x^\mu) = (1, -\omega\rho \sin \omega\tau^1, \omega\rho \cos \omega\tau^1, 0)$$
$$= (1, -\omega x^2, \omega x^1, 0) ,$$
$$(x'^\mu) = (0, \cos \omega\tau^1, \sin \omega\tau^1, 0)(1 - \omega^2\rho^2)^{-1/2} ,$$
$$= (0, x^1/\rho, x^2/\rho, 0)(1 - \omega^2\rho^2)^{-1/2} , \qquad (26)$$

giving $L_S = -\tfrac{1}{2}$. Because of eqs. (19) and (20b) we get

$$p^1_\mu = g_{\mu\nu} \dot x^\nu (1 - \omega^2\rho^2)^{-1} ,$$
$$p^2_\mu = -g_{\mu\nu} x'^\nu (1 - \omega^2\rho^2) . \qquad (27)$$

The canonical momenta (27) are "embedded" by the following solutions of the H–J equation (24):

$$S^1 = -\tau^1 + 2(1 - \omega^2\rho^2)^{-1}(x^0 - \omega\rho^2\theta) ,$$
$$S^2 = \tfrac{1}{2}\tau^2 - g(\rho) ,$$
$$\theta = \text{arctg}\,(x^2/x^1) , \quad dg/d\rho = (1 - \omega^2\rho^2)^{1/2} , \qquad (28)$$

which give

$$\partial_\mu S^1 = 2(1 - \omega^2\rho^2)^{-1} f_\mu(x) + h(x) \partial_\mu \rho ,$$
$$\partial_\mu S^2 = -(1 - \omega^2\rho^2)^{1/2} \partial_\mu \rho ,$$
$$(f_\mu(x)) = (1, \omega x^2, -\omega x^1, 0) ,$$
$$h(x) = 4\omega\rho(1 - \omega^2\rho^2)^{-2}(\omega x^0 - \theta) . \qquad (29)$$

On the extremal $\Sigma^{(2)}$ we have $x^0 = \tau^1$ and $\theta = \omega\tau^1$, i.e. on $\Sigma^{(2)}$ the functions

$$\psi^1_\mu = \partial_{(2)} S^2 \partial_\mu S^1 = \tfrac{1}{2} \partial_\mu S^1 ,$$
$$\psi^2_\mu = \partial_{(1)} S^1 \partial_\mu S^2 = -\partial_\mu S^2 , \qquad (30)$$

coincide with the canonical momenta (27).

Furthermore, for

$$S_{\mu\nu} := \partial_\mu S^1 \partial_\nu S^2 - \partial_\mu S^2 \partial_\nu S^1 ,$$

we get

$$(S_{01}, S_{02}, S_{03}) = -2(1 - \omega^2\rho^2)^{-1/2} e_\rho ,$$
$$(S_{23}, S_{31}, S_{12}) = -2(1 - \omega^2\rho^2)^{-1/2} \omega\rho\, e_3 . \qquad (31)$$

Notice that $S^1 = \tau^1$ on $\Sigma^{(2)}$!

For a given Hamilton function $H(x, p)$ the "velocities" v^μ_j obey the relation [13] [‡2]

‡2 The second half of the equations printed in ref. [13] is not correct (this was kindly pointed out to H.A. Kastrup by Prof. Géheniau, Dr. Biran and Dr. David from Brussels). The correct form will be contained in the review mentioned in ref. [7].

$$K^{-1}v_j^\mu T^j{}_k = \partial H/\partial p_\mu^k\,, \quad K := \Pi_\mu^j\, v_j^\mu - L\,. \tag{32}$$

Since $K = 3L_S$ and $T^j{}_k = \delta^j_k L_S$ in our case, we have

$$v_j^\mu = 3\partial H/\partial p_\mu^j\,, \tag{33}$$

implying

$$v_1^\mu = (4H^2)^{-1} g^{\mu\nu}[p_\nu^2(p^1\cdot p^2) - p_\nu^1(p^2)^2]\,,$$

$$v_2^\mu = (4H^2)^{-1} g^{\mu\nu}[p_\nu^1(p^1\cdot p^2) - p_\nu^2(p^1)^2]\,.$$

Inserting for p_μ^j the functions ψ_μ^j of eqs. (30), we get the "slope" functions

$$\phi_1^\mu(x,\tau) = g^{\mu\nu}f_\nu(x) = f^\mu(x)\,,$$

$$\phi_2^\mu(x,\tau) = -(1-\omega^2\rho^2)^{-1/2} g^{\mu\nu}[\tfrac{1}{2}h(x)f_\nu(x) + \partial_\nu\rho]\,, \tag{34}$$

which again coincide with the derivatives (26) on $\Sigma^{(2)}$ and which obey the integrability condition

$$\frac{\mathrm{d}}{\mathrm{d}\tau^2}\phi_1^\mu(x,\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau^1}\phi_2^\mu(x,\tau) \quad \text{or} \quad \phi_2^\nu\partial_\nu\phi_1^\mu = \phi_1^\nu\partial_\nu\phi_2^\mu$$

not only on $\Sigma^{(2)}$.

If we define an electromagnetic field associated with a given string motion by

$$F_{\mu\nu} = \lambda(\partial_\mu S^1\partial_\nu S^2 - \partial_\mu S^2\partial_\nu S^1) = \lambda S_{\mu\nu}\,, \quad \lambda = \text{const.}\,, \tag{35}$$

two remarks (at least) are in order: First, there is the question, whether it is always possible to find solutions $S^j(x,\tau)$ of the H–J equation (24a) which obey the transversality conditions (24b) for a given extremal $x^\mu(\tau)$ with canonical momenta p_μ^j. The answer is "yes" [14]. Second, the rhs of eq. (35) in general will depend on the parameters τ^j. These parameters are to be considered as constants, if we require the homogeneous Maxwell equations to hold, i.e. if

$$F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

is to be a closed form!

For the special motion discussed above the definition (35) according to eqs. (31) gives the fields

$$\boldsymbol{E} = -(F^{01},F^{02},F^{03}) = -2\lambda(1-\omega^2\rho^2)^{-1/2}\,\boldsymbol{e}_\rho\,,$$

$$\boldsymbol{B} = -(0,0,F^{12}) = 2\lambda(1-\omega^2\rho^2)^{-1/2}\,\omega\rho\,\boldsymbol{e}_3\,, \tag{36}$$

with the properties

$$\boldsymbol{E}^2 - \boldsymbol{B}^2 = 4\lambda^2\,, \quad \boldsymbol{E}\cdot\boldsymbol{B} = 0\,,$$

$$\operatorname{div}\boldsymbol{E} = -(2\lambda/\rho)(1-\omega^2\rho^2)^{-3/2}\,, \quad \operatorname{curl}\boldsymbol{E} = 0\,,$$

$$\operatorname{div}\boldsymbol{B} = 0\,, \quad \operatorname{curl}\boldsymbol{B} = -2\lambda\omega(1-\omega^2\rho^2)^{-3/2}\,\boldsymbol{e}_\theta\,.$$

On $\Sigma^{(2)}$ the Poynting vector $\boldsymbol{E}\times\boldsymbol{B} = 4\lambda^2(1-\omega^2\rho^2)^{-1} \times \omega\rho\boldsymbol{e}_\theta$ is – up to a constant – the spatial part of the canonical momentum (p_μ^1).

The fields (36) – and even more so the charge and current densities – have unpleasant singularities for $\rho = \omega^{-1}$, which makes an immediate physical interpretation difficult. In this respect the fields (5) look more decent!

In mechanics solutions of the H–J equation are also important in connection with conservation laws: If $S(t,q;a)$ is a solution of the H–J equation depending on the parameter a, then $(\partial S/\partial a)(t,q(t);a)$ is a constant of motion along an extremal $q(t)$ [15], for which $p_j(t) = \partial_j S(t,q(t);a)$. – Noether's theorem is a consequence of this property, if one interprets a as the parameter of the group $t\to\hat{t}(a)$, $q^j\to\hat{q}^j(a)$, $\hat{t}(a=0)=t$, $\hat{q}^j(0)=q^j$, which leaves $\mathrm{d}S(t,q) = L\,\mathrm{d}t$ invariant. The conserved quantity then becomes

$$\partial S(\hat{t},\hat{q})/\partial a|_{a=0} = \partial_t S T + \partial_j S Q^j = -HT + p_j Q^j\,,$$

$$T = \partial\hat{t}/\partial a|_{a=0}\,, \quad Q^j = \partial\hat{q}^j/\partial a|_{a=0}\,.$$

In field theories there exists a corresponding theorem ‡3: If $S^j(x,\tau)$, $j=1,2$, is a solution of Carathéodory's H–J equation which depends on a parameter a, then the current

$$G^j = (\partial S^k/\partial a)\,\overline{\Delta}^j{}_k\,, \quad j=1,2\,,$$

$$\Delta^j{}_k = \partial_{(k)}S^j + v_k^\mu\partial_\mu S^j\,, \tag{37}$$

is conserved "along" an extremal $\Sigma^{(2)}$, for which eqs. (24b) hold: $\partial_{(1)}G^1 + \partial_{(2)}G^2 = 0$! Here $\overline{\Delta}^j{}_k$ denotes the algebraic complement of $\Delta^j{}_k$!

For the solutions (28) we have on the extremal (25)

$$\Delta^1{}_1 = 1\,, \quad \Delta^1{}_2 = \Delta^2{}_1 = 0\,, \quad \Delta^2{}_2 = -\tfrac{1}{2}\,.$$

Taking $a = \omega$, we get the current

‡3 The first generalization to fields is due to Fréchet [16]. The general case was discussed by Dedecker, see ref. [12]. Details of the proof will be contained in the review by Kastrup mentioned in ref. [7].

$$G^1 = -\omega\rho^2(1-\omega^2\rho^2)^{-1}\tau^1 ,$$

$$G^2 = \omega \int^{\rho} d\bar{\rho}\, \bar{\rho}^2(1-\omega^2\bar{\rho}^2)^{-1/2} , \tag{38}$$

for which

$$\partial_{(1)}G^1 = -\omega\rho^2(1-\omega^2\rho^2)^{-1} ,$$

$$\partial_{(2)}G^2 = \omega\rho^2(1-\omega^2\rho^2)^{-1/2}\, d\rho/d\tau^2$$

$$= \omega\rho^2(1-\omega^2\rho^2)^{-1} ,$$

i.e. the current is indeed conserved! This example – which itself does not seem to be a very interesting one – indicates the intriguing new possibilities of H–J theories for fields.

References

[1] Y. Nambu, in: Symmetries and quark models, ed. R. Chand (Gordon and Breach, New York, 1970) p. 269;
C. Rebbi, Phys. Rep. 12C (1974) 1;
J. Scherk, Rev. Mod. Phys. 47 (1975) 123.

[2] M. Lüscher, K. Symanzik and P. Weisz, Nucl. Phys. B173 (1980) 365;
A.M. Polyakov, Nucl. Phys. B164 (1980) 171;
J.L. Gervais and A. Neveu, Nucl. Phys. B163 (1980) 189;
A.A. Migdal, Phys. Lett. 96B (1980) 333;
and further references in these papers.

[3] M. Kalb and P. Ramond, Phys. Rev. D9 (1974) 2273;
Y. Nambu, Phys. Rev. D10 (1974) 4262;
F. Lund and T. Regge, Phys. Rev. D14 (1976) 1524;
F. Englert and P. Windey, Nucl. Phys. B135 (1978) 529;
Y. Nambu, in: Quark confinement and field theory, eds. D.R. Stump and D.H. Weingarten (Wiley, New York, 1977) p. 1.

[4] H.A. Kastrup, Phys. Lett. 78B (1978) 433; 82B (1979) 237.

[5] Y. Choquet-Bruhat, C. Dewitt-Morette and M. Dillard-Bleick, Analysis, manifolds and physics (North-Holland, Amsterdam, 1977) p. 251.

[6] M. Rinke, Comm. Math. Phys. 73 (1980) 265.

[7] C. Carathéodory, Acta Sci. Math. (Szeged) 4 (1929) 193;
reprinted in: C. Carathéodory, Gesammelte Mathematische Schriften, Bd. 1 (Beck, München, 1954) p. 401;
H.A. Kastrup, Phys. Lett. 70B (1977) 195.

[8] H.A. Kastrup, Canonical theories for dynamical systems in physics, in preparation.

[9] Y. Nambu, Phys. Lett. 92B (1980) 327;
see also: T. Eguchi, Phys. Rev. Lett. 44 (1980) 126.

[10] A. Schild, Phys. Rev. D16 (1977) 1722.

[11] C. Godbillon, Géométrie différentielle et méchanique analytique (Hermann, Paris, 1969) Chs. I, V, VI;
Y. Choquet-Bruhat, C. Dewitt-Morette and M. Dillard-Bleick, Analysis, manifolds and physics (North-Holland, Amsterdam, 1977) Ch. IV.

[12] Th. Lepage, Acad. Roy. Belg. Bull. Cl. Sci. (5e Sér.) 22 (1936) 716, 1036; 27 (1941) 27; 28 (1942) 73, 247;
see also: P. Dedecker, in: Proc. Coll. Intern. de Géométrie différentielle (Strasbourg, 1953) (CNRS, Paris, 1953) p. 17.

[13] H.A. Kastrup, Phys. Lett. 70B (1977) 195, eqs. (7) (first half).

[14] H. Boerner, Math. Ann. (Berlin) 112 (1936) 187;
E. Hölder, Jahresb. Deutsch. Math.-Verein. (Stuttgart) 49 (1939) 162;
Th. Lepage, Acad. Roy. Belg. Bull. Cl. Sci. (5e Sér.) 28 (1942) 73, 247.
L. Van Hove, Acad. Roy. Belg. Bull. Cl. Sci. (5e Sér.) 31 (1945) 625.

[15] E.T. Whittaker, A treatise on the analytical dynamics of particles and rigid bodies, 4th ed. (Cambridge U.P., Cambridge, 1959) p. 324;
I.M. Gel'fand and S.V. Fomin, Calculus of variations (Prentice-Hall, Englewood Cliffs, 1963) p. 90 (Theorem 1).

[16] M. Fréchet, Ann. Mat. Pura Appl. 11 (1905) 187;
see also: Th. De Donder, Théorie invariantive du calcul des variations, Nouv. éd. (Gauthier-Villars, Paris, 1935) Ch. 50.