The next-to-next-to-leading QCD calculation of the moments of deep inelastic structure functions

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Abstract: We present the calculation of the analytic next-to-next-to-leading perturbative QCD corrections in the leading twist approximation for the moments N=2,4,6,8 of the flavour singlet deep inelastic structure functions F_2 and F_L. We calculate the three-loop anomalous dimensions of the corresponding singlet operators and the three-loop coefficient functions of the structure functions F_L and F_2. In addition, we obtained the 10^{th} moment for the non-singlet structure functions in the same order of perturbative QCD.

1 Introduction

The calculation of the next-to-next-to-leading (NNL) QCD approximation for the structure functions F_2 and F_L of deep inelastic electron-nucleon scattering is important for the understanding of perturbative QCD and for an accurate comparison of perturbative QCD with experiment. To obtain the NNL approximation for these structure functions in the operator product expansion (OPE) formalism one needs the 3-loop anomalous dimensions of the operators, the 2-loop Wilson coefficient functions for F_2 and the 3-loop coefficient functions for F_L. At present, these structure functions are known in the next-to-leading approximation only, since the 3-loop anomalous dimensions and the 3-loop coefficient functions for F_L were not calculated yet.

The 1-loop anomalous dimensions were calculated in Ref. [1]. The complete 1-loop coefficient functions were obtained in Ref. [2] (see also the references therein). Anomalous dimensions in the 2-loop order were obtained in Refs. [3]-[6] and the 2-loop coefficient functions were calculated in Refs. [7]-[11]. In Ref. [12] we presented the NNL corrections of the *non-singlet* type in the leading twist approximation for the moments N=2,4,6,8 of the deep inelastic structure functions F_2 and F_L.

In the present contribution we describe the more recent calculation of the NNL QCD corrections to the *singlet* moments N=2,4,6,8 of both structure functions F_2 and F_L. To this end, we calculated the corresponding 3-loop anomalous dimensions and the 3-loop coefficient functions for the structure function F_L. In addition, we obtained the 3-loop coefficient functions for the

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structure function F_2 (for N=2,4,6,8) and we obtained the N=10 *non-singlet* moments of F_2 and F_L. The calculations are done for the leading twist approximation for zero quark masses. The complete set of the results of our calculations can be found in Ref. [13] where further details and an analysis of the results are given.

2 The calculation

We need to calculate the hadronic part of the amplitude for unpolarized deep inelastic electron-nucleon scattering which is given by the hadronic tensor

$$
\begin{aligned}
W_{\mu\nu}(p,q) &= \frac{1}{4\pi}\int d^4z e^{iq\cdot z}\langle p,\mathrm{nucl}|J_\mu(z)J_\nu(0)|\,\mathrm{nucl},p\rangle \\
&= e_{\mu\nu}\frac{1}{2x}F_L(x,Q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,Q^2) \\
e_{\mu\nu} &= \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \\
d_{\mu\nu} &= \left(-g_{\mu\nu} - p_\mu p_\nu\frac{4x^2}{q^2} - (p_\mu q_\nu + p_\nu q_\mu)\frac{2x}{q^2}\right),
\end{aligned} \tag{1}
$$

where J_μ is the electromagnetic quark current, $x = Q^2/(2p\cdot q)$ is the Bjorken scaling variable $(0 < x \leq 1)$, $Q^2 = -q^2$ is the transferred momentum and $|\,\mathrm{nucl},p\rangle$ is the nucleon state with momentum p. Spin averaging is assumed. The longitudinal structure function F_L is related to the structure function F_1 by $F_L = F_2 - 2xF_1$.

As one approaches the Bjorken limit, $Q^2 \to \infty$, x fixed, one can show that the integration region in Eq. (1) near the light cone $z^2 \approx 0$ progressively dominates[14], due to increasingly rapid phase fluctuations of the term $e^{iq\cdot z}$ outside the light cone region (and presuming that the integrand $\langle p,\mathrm{nucl}|J_\mu(z)J_\nu(0)|\,\mathrm{nucl},p\rangle$ varies smoothly outside the light cone). Since we have to deal with this non-local limit $z^2 \approx 0$, a formal operator product expansion in terms of local operators can only be applied together with the dispersion relation technique [15]. These techniques together provide a systematic way to study[2] the leading and non-leading contributions to the hadronic tensor.

The tensor $W_{\mu\nu}$ is, by application of the optical theorem, related to a scattering amplitude $T_{\mu\nu}$ which is a more convenient quantity for practical calculations since it has a time ordered product of currents to which standard perturbation theory applies ($T_{\mu\nu}$ is the amplitude for forward elastic photon–nucleon scattering)

$$
W_{\mu\nu}(p,q) = \frac{1}{2\pi}\mathrm{Im}\,T_{\mu\nu}(p,q), \qquad T_{\mu\nu}(p,q) = i\int d^4z e^{iqz}\langle p,\mathrm{nucl}|T\left(J_\mu(z)J_\nu(0)\right)|\,\mathrm{nucl},p\rangle. \tag{2}
$$

The operator product expansion in terms of local operators for a time ordered product of the two electromagnetic hadronic currents reads

$$
i\int d^4z e^{iqz}T\left(J_{\nu_1}(z)J_{\nu_2}(0)\right) =
$$

$$
\sum_{N,j}(\frac{1}{Q^2})^N\left[\left(g_{\nu_1\nu_2} - \frac{q_{\nu_1}q_{\nu_2}}{q^2}\right)q_{\mu_1}q_{\mu_2}C^j_{L,N}(\frac{Q^2}{\mu^2},a_s) - \left(g_{\nu_1\mu_1}g_{\nu_2\mu_2}q^2 - g_{\nu_1\mu_1}q_{\nu_2}q_{\mu_2}\right.\right.
$$

[2]For reviews see Refs. [16, 17, 18].

$$-g_{\nu_2\mu_2}q_{\nu_1}q_{\mu_1}+g_{\nu_1\nu_2}q_{\mu_1}q_{\mu_2}\big)\, C^j_{2,N}(\frac{Q^2}{\mu^2},a_s)\Big]\, q_{\mu_3}...q_{\mu_N}O^{j,\{\mu_1,...,\mu_N\}}(0) + \text{ higher twists,}$$

$$\text{j}=\alpha,\psi,\text{G} \qquad (3)$$

where everything is assumed to be renormalized (with μ being the renormalization scale). The use of the OPE in the short distance regime ($z \to 0$) differs from its use in the light cone region. In the former case the sum over spin-N extends to a finite value for a given approximation, while in the latter (the one we have to deal with) the sum over N extends to infinity. The sum over N runs over the standard set of the spin-N twist-2 irreducible (i.e. symmetrical and traceless in the indices $\mu_1,\cdots,\mu_N$) flavour non-singlet quark operators and the singlet quark and gluon operators:

$$O^{\alpha,\{\mu_1,\cdots,\mu_N\}} = \overline{\psi}\lambda^\alpha\gamma^{\{\mu_1}D^{\mu_2}\cdots D^{\mu_N\}}\psi, \quad \alpha=1,2,...,(n_f^2-1) \qquad (4)$$

$$O^{\psi,\{\mu_1,\cdots,\mu_N\}} = \overline{\psi}\gamma^{\{\mu_1}D^{\mu_2}\cdots D^{\mu_N\}}\psi, \qquad (5)$$

$$O^{G,\{\mu_1,\cdots,\mu_N\}} = G^{\{\mu\mu_1}D^{\mu_2}\cdots D^{\mu_{N-1}}G^{\mu_N\mu\}}. \qquad (6)$$

Here and in the following we denote the generators of the flavour group $SU(n_f)$ by λ^α, and the covariant derivative by D^{μ_i}; in addition, it is understood that the symmetrical and traceless part is taken with respect to the indices in curly brackets. The functions $C^j_{k,N}(Q^2/\mu^2,a_s)$ are the coefficient functions for the above operators. Since the coefficient functions $C^\alpha_{k,N}$ of non-singlet operators depend trivially on the number α (see e.g. Ref. [12]) we will use for them the standard notation $C^{ns}_{k,N}$. Here and throughout the whole contribution we use the notation

$$a_s = \frac{g^2}{16\pi^2} = \frac{\alpha_s}{4\pi} \qquad (7)$$

for the QCD strong coupling constant. The direct application of the OPE of Eq. (3) to the Green function $T_{\mu\nu}$ leads to a formal expansion for $T_{\mu\nu}$ in terms of the variable $q\cdot p/Q^2 = 1/(2x)$ i.e. an expansion for unphysical $x\to\infty$,

$$i\int d^4z e^{iqz}\langle p,\text{nucl}|T\left(J_\mu(z)J_\nu(0)\right)|\text{ nucl},p\rangle =$$

$$\sum_{N,j}(\frac{1}{2x})^N\left[\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)C^j_{L,N}\left(\frac{Q^2}{\mu^2},a_s\right)-\left(g_{\mu\nu}+p_\mu p_\nu\frac{4x^2}{q^2}\right.\right.$$

$$\left.\left.+(p_\mu q_\nu+p_\nu q_\mu)\frac{2x}{q^2}\right)C^j_{2,N}\left(\frac{Q^2}{\mu^2},a_s\right)\right]A^j_{\text{nucl},N}(m_n^2/\mu^2) + \text{ higher twists,}$$

$$\text{j}=ns,\psi,\text{G} \qquad (8)$$

where the spin averaged matrix elements are defined as

$$\langle p,\text{nucl}|O^{j,\{\mu_1,...,\mu_N\}}|\text{nucl},p\rangle = p^{\{\mu_1}...p^{\mu_N\}}A^j_{\text{nucl},N}(m_n^2/\mu^2) \qquad (9)$$

and m_n is the nucleon mass.

To perform the proper analytic continuation of the representation Eq. (8) to the physical region $0<x\le 1$ one applies a dispersion relation in the complex x plane to the Green function $T_{\mu\nu}$. For electron-nucleon scattering where we have hermitian currents J^μ one finds that the

Mellin moments of the structure functions F_k are expressed through the parameters of the operator product expansion (3)

$$\frac{1+(-1)^N}{2} M_{k,N} \equiv \frac{1+(-1)^N}{2} \int_0^1 dx\, x^{N-2} F_k(x,Q^2) = \sum_{i=ns,\psi,G} C^i_{k,N}\left(\frac{Q^2}{\mu^2}, a_s\right) A^i_{\mathrm{nucl},N} \qquad (10)$$

Please note that the odd Mellin moments of F_k are not fixed by this equation. However, all moments in the complex N plane are fixed by analytic continuation from the even Mellin moments when all the even moments are known. This means that the structure functions in x-space, $0 < x \leq 1$, can be found by means of the inverse Mellin transformation when the (infinite set of) even moments are known.

The Q^2-dependence of the coefficient functions can be studied by the use of the renormalization group equations

$$\left[\mu^2\frac{\partial}{\partial\mu^2} + \beta(a_s(\mu^2))\frac{\partial}{\partial a_s(\mu^2)}\right] C^i_{k,N}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) = C^j_{k,N}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right)\gamma^{ji}_N(a_s(\mu^2)) \quad ,\ \mathrm{i,j}=\psi,\mathrm{G} \qquad (11)$$

$$\left[\mu^2\frac{\partial}{\partial\mu^2} + \beta(a_s(\mu^2))\frac{\partial}{\partial a_s(\mu^2)}\right] C^{ns}_{k,N}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) = C^{ns}_{k,N}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right)\gamma^{ns}_N(a_s(\mu^2)) \qquad (12)$$

where Eq. (11) represents the singlet sector where quark and gluon operators mix under renormalization, and Eq. (12) is the non-singlet equation. $\beta(a)$ is the beta-function that determines the renormalization scale dependence of the renormalized coupling constant. It is known at three loops [19] in the $\overline{\mathrm{MS}}$ scheme

$$\begin{aligned}
\frac{\partial a_s}{\partial \ln\mu^2} &\equiv \beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 + O(a_s^5), \\
\beta_0 &= \left(\frac{11}{3}C_A - \frac{4}{3}T_F n_f\right) \\
\beta_1 &= \left(\frac{34}{3}C_A^2 - 4C_F T_F n_f - \frac{20}{3}C_A T_F n_f\right) \\
\beta_2 &= \left(\frac{2857}{54}C_A^3 + 2C_F^2 T_F n_f - \frac{205}{9}C_F C_A T_F n_f - \right. \\
&\quad \left. -\frac{1415}{27}C_A^2 T_F n_f + \frac{44}{9}C_F T_F^2 n_f^2 + \frac{158}{27}C_A T_F^2 n_f^2\right)
\end{aligned} \qquad (13)$$

where $C_F = \frac{4}{3}$ and $C_A = 3$ are the Casimir operators of the fundamental and adjoint representations of the colour group $SU(3)$, $T_F = \frac{1}{2}$ is the trace normalization of the fundamental representation and n_f is the number of (active) quark flavours. The anomalous dimensions $\gamma_N(a_s)$ determine the renormalization scale dependence of the operators, that is

$$\frac{d}{d\ln\mu^2} O_R^{j,\{\mu_1,\dots,\mu_N\}} \equiv -\gamma^{ji}_N(a_s)\, O_R^{i,\{\mu_1,\dots,\mu_N\}}, \qquad \mathrm{i,j}=\psi,\mathrm{G} \qquad (14)$$

$$\frac{d}{d\ln\mu^2} O_R^{ns,\{\mu_1,\dots,\mu_N\}} \equiv -\gamma^{ns}_N(a_s)\, O_R^{ns,\{\mu_1,\dots,\mu_N\}}. \qquad (15)$$

We define renormalized operators in terms of bare operators as $O_R = Z\, O_B$ and find

$$\frac{d}{d\ln\mu^2} O_R = \left(\frac{d}{d\ln\mu^2} Z\right) O_B = \left(\frac{d}{d\ln\mu^2} Z\right) Z^{-1} O_R \quad \Rightarrow \quad \gamma = -\left(\frac{d}{d\ln\mu^2} Z\right) Z^{-1} \qquad (16)$$

where it is understood that in the singlet case Z represents a matrix Z^{ij}. The renormalization group equations are solved in the standard form

$$C^{ns}_{k,N}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) \quad = \quad C^{ns}_{k,N}(1, a_s(Q^2)) \times \exp\left(-\int_{a_s(\mu^2)}^{a_s(Q^2)} da'_s\, \frac{\gamma^{ns}_N(a'_s)}{\beta(a'_s)}\right) \tag{17}$$

The solution for the singlet equations has a similar form but since one gets the exponential of a matrix of anomalous dimensions one has to define the exponential properly in the singlet case (i.e. a T-ordered exponential, see e.g. Ref. [20]). Here $a_s(Q^2) \equiv a_s(Q^2/\Lambda^2_{\overline{\rm MS}})$ is the renormalized (i.e. running) coupling constant at the renormalization scale Q^2 and $\Lambda_{\overline{\rm MS}}$ is the fundamental scale of QCD in the $\overline{\rm MS}$-scheme. In practice one may use the DGLAP evolution equations [21] for matrix elements of operators at the scale $\mu^2 = Q^2$ (i.e. Q^2-dependent parton distributions) instead of the renormalization group equations for the coefficient functions (for perturbative solutions of the DGLAP equations in moment space see e.g. Refs. [3, 22]).

We will now discuss the method [23] for the calculation of anomalous dimensions and coefficient functions in considerable detail as it applies to the singlet sector. Let us first elaborate on some details specific to the dimensional regularization[24] and the minimal subtraction scheme [25], and it's standard modification, the $\overline{\rm MS}$-scheme [2], which form a modern basis for multi-loop calculations in QCD. We use the symbol a_s for the renormalized coupling constant and a_b for the bare coupling constant. Although renormalization constants Z contain poles in ε in D= $4 - 2\varepsilon$ dimensions, anomalous dimensions are finite as D$\rightarrow$ 4. This fact gives expressions for the higher poles of Z in terms of the first poles of Z. To see this we write Eq. (16) as

$$\gamma Z = -\left(\frac{d}{d\ln\mu^2} Z(a_s, \frac{1}{\varepsilon})\right) = -\left(\frac{\partial}{\partial a_s} Z\right)\frac{da_s}{d\ln\mu^2} = -\left(\frac{\partial}{\partial a_s} Z\right)[-\varepsilon a_s + \beta(a_s)] \tag{18}$$

where $\beta(a_s)$ is the 4-dimensional beta function of Eq. (13) and $[-\varepsilon a + \beta(a)]$ is the beta function in $4-2\varepsilon$ dimensions. This latter function receives no higher order corrections in ε due to the form of renormalization factors in the minimal subtraction scheme. The factors Z^{ij} are calculated as series in a_s, and have the well known form

$$Z^{ij} = Z^{ij(0)} + Z^{ij(1)}(a_s)/\varepsilon + Z^{ij(2)}(a_s)/\varepsilon^2 + \cdots, \qquad i, j = \psi, G$$

with $Z^{\psi\psi(0)} = Z^{GG(0)} = 1$, $Z^{\psi G(0)} = Z^{G\psi(0)} = 0$. ¿From equation (18) and the form of the factors Z^{ij} it follows that the anomalous dimensions are expressed through the coefficients in front of the first poles of Z as

$$\gamma^{ij} = a_s\left(\frac{\partial}{\partial a_s} Z^{ij(1)}(a_s)\right) \qquad i, j = \psi, G \tag{19}$$

where $Z^{ij(1)}(a_s)$ was defined as the order $1/\varepsilon$ part of Z^{ij}. The coefficients of higher poles in Z can then be expressed in terms of γ^{ij} by substituting the expression for γ^{ij} back into equation (18).

The operator product expansion of Eq. (3) is an operator statement and both the coefficient functions $C^i_{k,N}$ and the anomalous dimensions γ^{ij}_N of the operators are functions and do therefore not depend on the hadronic states of the Green function to which one wishes to apply the OPE. The information on the hadronic target is contained in the operator matrix elements A^i_N in Eq. (9) which are generally not calculable perturbatively. It is therefore standard to consider simpler Green functions with quarks and gluons as external particles, instead of the physical nucleon states, in the calculation of coefficient functions and anomalous dimensions. In this case the Green functions can be calculated in perturbation theory as well as the operator matrix elements and the anomalous dimensions and coefficient functions can be extracted as will be shown below in detail.

Let us consider the following 4-point Green functions

$$T_{\mu\nu}^{\mathrm{q}\gamma\mathrm{q}\gamma}(p,q) = i\int d^4z e^{iqz}\langle p, \mathrm{quark}|T\left(J_\mu(z)J_\nu(0)\right)|\ \mathrm{quark}, p\rangle \tag{20}$$

$$T_{\mu\nu}^{\mathrm{g}\gamma\mathrm{g}\gamma}(p,q) = i\int d^4z e^{iqz}\langle p, \mathrm{gluon}|T\left(J_\mu(z)J_\nu(0)\right)|\ \mathrm{gluon}, p\rangle \tag{21}$$

where the label γ is used to indicate an external photon, q indicates an external quark and g an external gluon. Spin and colour averaging for the quark and gluon states is assumed. Analogously to the decomposition of the hadronic tensor $W_{\mu\nu}$ in terms of F_2 and F_L we decompose the Green functions $T_{\mu\nu}$ in terms of T_L and T_2. Applying the OPE to $T^{\mathrm{q}\gamma\mathrm{q}\gamma}$ and $T^{\mathrm{g}\gamma\mathrm{g}\gamma}$ we find the following equations for the renormalized Green functions

$$\begin{aligned}
&T_k^{\mathrm{q}\gamma\mathrm{q}\gamma}(p,q,a_s,\mu^2,\varepsilon) = \\
&\sum_{N=2}^{\infty}\left(\frac{1}{2x}\right)^N\Bigg[\left(C_{k,N}^{\psi}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{\psi\psi}(a_s,\frac{1}{\varepsilon}) + C_{k,N}^{G}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{G\psi}(a_s,\frac{1}{\varepsilon})\right)A_{\mathrm{quark},N}^{\psi}(a_s,\frac{p^2}{\mu^2},\varepsilon) \\
&+\left(C_{k,N}^{\psi}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{\psi G}(a_s,\frac{1}{\varepsilon}) + C_{k,N}^{G}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{GG}(a_s,\frac{1}{\varepsilon})\right)A_{\mathrm{quark},N}^{G}(a_s,\frac{p^2}{\mu^2},\varepsilon)\Bigg] + O(p^2) \quad (22)
\end{aligned}$$

$$\begin{aligned}
&T_k^{\mathrm{g}\gamma\mathrm{g}\gamma}(p,q,a_s,\mu^2,\varepsilon) = \\
&\sum_{N=2}^{\infty}\left(\frac{1}{2x}\right)^N\Bigg[\left(C_{k,N}^{\psi}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{\psi G}(a_s,\frac{1}{\varepsilon}) + C_{k,N}^{G}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{GG}(a_s,\frac{1}{\varepsilon})\right)A_{\mathrm{gluon},N}^{G}(a_s,\frac{p^2}{\mu^2},\varepsilon) \\
&+\left(C_{k,N}^{\psi}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{\psi\psi}(a_s,\frac{1}{\varepsilon}) + C_{k,N}^{G}(a_s,\frac{Q^2}{\mu^2},\varepsilon)Z_N^{G\psi}(a_s,\frac{1}{\varepsilon})\right)A_{\mathrm{gluon},N}^{\psi}(a_s,\frac{p^2}{\mu^2},\varepsilon)\Bigg] + O(p^2) \quad (23)
\end{aligned}$$

where k =2,L, $a_s \equiv a_s(\mu^2/\Lambda^2)$ and it is understood that the l.h.s. is renormalized by substituting the bare coupling constant in terms of the renormalized one,

$$a_b = a_s - \frac{\beta_0}{\varepsilon}a_s^2 + (\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon})a_s^3 + O(a_s^4) \tag{24}$$

The terms $O(p^2)$ in the r.h.s. of Eqs. (22) and (23) indicate higher twist contributions. The renomalization factors for the external quark and gluon lines are overall factors on both sides of the equations and are omited. The coefficient functions on the r.h.s are renormalized quantities. The matrix elements A_N^i are the matrix elements of *bare* operators and are defined as in Eq. (9) with the nucleon states replaced by the appropriate quark or gluon states.

It is known that the gauge invariant operators O^ψ and O^G mix under renormalization with unphysical operators (that are BRST variations of some operators or that vanish by the equations of motion) [18, 6, 26]. But physical matrix elements (i.e. on-shell matrix elements with physical polarizations) of such unphysical operators vanish. Since the method that is described below deals with physical matrix elements we omited the unphysical operators in Eqs. (22) and (23).

Starting from Eqs. (22) and (23), the anomalous dimensions and the coefficient functions are calculated using the method of projections of Ref. [23]. It reduces the calculation of (moments of) coefficient functions and anomalous dimensions to the calculation of diagrams of the propagator type instead of the 4-point diagrams that contribute to $T_{\mu\nu}$. This method relies heavily on the use of dimensional regularization and the minimal subtraction scheme and implicitly involves a considerable rearrangement of infrared and ultraviolet divergences.

The method consists of applying the following projection operator to both sides of Eqs. (22) and (23).

$$\mathcal{P}_N \equiv \left[\frac{q^{\{\mu_1} \cdots q^{\mu_N\}}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \cdots \partial p^{\mu_N}} \right] \Bigg|_{p=0} \tag{25}$$

Here $q^{\{\mu_1} \cdots q^{\mu_N\}}$ is the harmonic (i.e. symmetrical and traceless) part of the tensor $q^{\mu_1} \cdots q^{\mu_N}$ (see next section). The operator $\mathcal{P}_N$ is applied to the integrands of all Feynman diagrams (nullifying p before taking the limit $\varepsilon \to 0$, to dimensionally regularize the infrared divergences as $p \to 0$ for individual diagrams). It is important to realize that this operation does not act on the renormalization constants Z_N^{ij} and the coefficient functions on the r.h.s. of Eqs. (22), (23). It does however act on the matrix elements A_N^i. The nullification of p has the effect that of all the diagrams that contribute to the perturbative expansion of A_N^i only the tree level terms (i.e. with no loops) survive since all diagrams containing loops become massless tadpole diagrams. Massless tadpole diagrams are put to zero in dimensional regularization. Furthermore, the $N^{\rm th}$ order differentiation in the operator $\mathcal{P}_N$ has the effect that $\mathcal{P}_N$ projects out only the N^{th} moment since of all the factors $1/(2x)^{N'}$ only $1/(2x)^N$ gives a non zero contribution after nullifying p. On the left hand side the effect of $\mathcal{P}_N$ is to effectively reduce the 4-point diagrams that contribute to $T_{\mu\nu}$ to 2-point diagrams (this follows from the nullification of the momentum p), which drastically simplifies the calculation. We apply the operator $\mathcal{P}_N$ after the tensor structures $2, L$ have been projected out because the operator $\mathcal{P}_N$ would mutilate the tensor structure of $T_{\mu\nu}$. In the projector $\mathcal{P}_N$ we use the harmonic tensor $q^{\{\mu_1} \cdots q^{\mu_N\}}$ to remove higher twist contributions (the $O(p^2)$ terms in Eqs. (22) and (23)) that after differentiation with respect to p^μ survive as terms proportional to the metric tensor.

After application of the projection operator $\mathcal{P}_N$ to Eqs. (22) and (23) we have

$$T_{k,N}^{\mathrm{q}\gamma\mathrm{q}\gamma}(\frac{Q^2}{\mu^2}, a_s, \varepsilon) = \left(C_{k,N}^{\psi}(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{\psi\psi}(a_s, \frac{1}{\varepsilon}) + C_{k,N}^{G}(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{G\psi}(a_s, \frac{1}{\varepsilon}) \right) A_{\mathbf{quark},N}^{\psi,\mathbf{tree}}(\varepsilon) \tag{26}$$

$$T_{k,N}^{\mathrm{g}\gamma\mathrm{g}\gamma}(\frac{Q^2}{\mu^2}, a_s, \varepsilon) = \left(C_{k,N}^{\psi}(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{\psi G}(a_s, \frac{1}{\varepsilon}) + C_{k,N}^{G}(a_s, \frac{Q^2}{\mu^2}, \varepsilon) Z_N^{GG}(a_s, \frac{1}{\varepsilon}) \right) A_{\mathbf{gluon},N}^{G,\mathbf{tree}}(\varepsilon) \tag{27}$$

where $k = 2, L$ and we defined

$$T_{k,N}(\frac{Q^2}{\mu^2}, a_s, \varepsilon) \equiv \frac{q^{\{\mu_1} \cdots q^{\mu_N\}}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \cdots \partial p^{\mu_N}} T_k(p, q, a_s, \mu^2, \varepsilon) \Bigg|_{p=0}. \tag{28}$$

It should be understood that (26) and (27) represent a large coupled system of equations when both sides are expanded in powers of a_s and ε (i.e. C is expanded in positive powers of ε and Z is expanded in negative powers of ε).

After the calculation of $T^{\mathrm{q}\gamma\mathrm{q}\gamma}$ and $T^{\mathrm{g}\gamma\mathrm{g}\gamma}$ in the order a_s^3 and the determination of the tree level matrix elements $A_N^{j,\mathrm{tree}}$ one can solve Eqs. (26) and (27) simultaneously to obtain C_k^{ψ}, C_k^G, $Z^{\psi\psi}$ and $Z^{\psi G}$ in order a_s^3 but, unfortunately, $Z^{G\psi}$ and Z^{GG} only in the order a_s^2. This limitation follows directly from the fact that C_k^ψ starts from the order a_s^0 but C_k^G starts from order a_s since the photon couples *directly* only to quarks. In solving the equations it is essential that all poles of the Z factors are fully expressed in terms of the anomalous dimensions as was discussed in the beginning of this section. Coefficient functions and operator matrix elements are finite as $\varepsilon \to 0$ but one must make sure that sufficiently high powers in ε are taken into account. For example, one should consider order ε^2 contributions for C_k^ψ at order a_s. We stress that by calculating only propagator type diagrams in the l.h.s. of Eqs. (26) and (27) we can get both renormalization constants of operators and coefficient functions.

To obtain $Z^{G\psi}$ and Z^{GG} in order a_s^3 we calculated two more unphysical Green functions $T^{\mathrm{q}\phi\mathrm{q}\phi}$ and $T^{\mathrm{g}\phi\mathrm{g}\phi}$, in which the photon is replaced by an external scalar particle ϕ that couples *directly* only to gluons. The vertices that describe the coupling between the external scalar field ϕ and the gluons follow from adding the simplest gauge invariant interaction term $\phi G_{\mu\nu}^a G_a^{\mu\nu}$ (where $G_{\mu\nu}^a$ is the QCD field strength tensor) to the QCD Lagrangian. For the Green functions $T^{\mathrm{q}\phi\mathrm{q}\phi}$ and $T^{\mathrm{g}\phi\mathrm{g}\phi}$ an OPE similar to (3) exists with the same operators but with different coefficient functions C_ϕ^G and C_ϕ^ψ, where C_ϕ^ψ starts from the order a_s and C_ϕ^G starts from the order a_s^0.

One can repeat the steps that led to Eqs. (26) and (27) for the Green function $T^{\mathrm{q}\phi\mathrm{q}\phi}$ and $T^{\mathrm{g}\phi\mathrm{g}\phi}$ but in addition, the external operators $G_{\mu\nu}^a G_a^{\mu\nu}$ have to be renormalized. After the calculation of $T_N^{\mathrm{q}\phi\mathrm{q}\phi}$ and $T_N^{\mathrm{g}\phi\mathrm{g}\phi}$ in the order a_s^3 one can obtain C_ϕ^G, C_ϕ^ψ, $Z^{G\psi}$ and Z^{GG} in the order a_s^3. Please note that the coefficients C_ϕ^G and C_ϕ^ψ are obtained as a byproduct and are not important for the physical process under consideration.

Summarizing, we apply the operator $\mathcal{P}_N$ for N=2,4,6,8 to 4 different Green functions, $T_k^{\mathrm{q}\gamma\mathrm{q}\gamma}$, $T_k^{\mathrm{g}\gamma\mathrm{g}\gamma}$, $T^{\mathrm{q}\phi\mathrm{q}\phi}$ and $T^{\mathrm{g}\phi\mathrm{g}\phi}$ and we sum over the physical spin polarizations of the external quarks and gluons. For the external quarks (in $T^{\mathrm{q}\gamma\mathrm{q}\gamma}$ and $T^{\mathrm{q}\phi\mathrm{q}\phi}$) the sum over the polarizations is performed by inserting the projection operator $\not{p}$ between the external quark legs and taking the trace over the strings of gamma matrices. For the external gluons (in $T_k^{\mathrm{g}\gamma\mathrm{g}\gamma}$ and $T^{\mathrm{g}\phi\mathrm{g}\phi}$) the sum over physical spins can be done by contracting the external gluon lines with $-g^{\alpha\beta}+(p^\alpha q^\beta + p^\beta q^\alpha)/p\cdot q - p^\alpha p^\beta q^2/(p\cdot q)^2$ in which the (on-shell) gluon has momentum p (with $p^2=0$). The presence of the extra powers of p poses considerable efficiency problems (the operator $\mathcal{P}_N$ will generate more than 3 times larger intermediate expressions as compared to the case of a simpler $g^{\alpha\beta}$ projection). Alternatively one may take the sum over physical gluon spins by contracting the external gluon lines with only $-g^{\alpha\beta}$ and adding external ghost contributions to the Green functions, $T_k^{\mathrm{h}\gamma\mathrm{h}\gamma}$ to $T_k^{\mathrm{g}\gamma\mathrm{g}\gamma}$ and $T^{\mathrm{h}\phi\mathrm{h}\phi}$ to $T^{\mathrm{g}\phi\mathrm{g}\phi}$, where the label h indicates an external ghost line. This procedure is identical to the standard use of ghost diagrams to remove unphysical polarizations of gluon propagators in the covariant gauge, and the ghost particle h is the same ghost that we use in closed loops. Although we now have to consider all diagrams that contribute to $T_k^{h\gamma h\gamma}$ and $T^{\mathrm{h}\phi\mathrm{h}\phi}$, and increase the total number of diagrams that we have to calculate, it still makes the computations more than a factor of 3 faster (since ghost diagrams are of a far simpler nature than gluon diagrams). We checked for the lowest moments that the two methods for taking the sum over the gluon spin polarizations gave the same results, but for the higher moments we only applied the ghost method.

The Feynman diagrams that we need to calculate are generated automatically with a special version of the diagram generator QGRAF[27]. The use of a Feynman diagram generator is very important since in total we need to calculate more than 10.000 3-loop diagrams for each value for the moment index N. For every class qγqγ, qϕqϕ etc. the full set of diagrams is put into a single file using a dedicated database program MINOS that manages information about thousands of diagrams and can be instructed to call other programs, giving them the proper information from a database file.

For the actual calculations we use programs written for the symbolic manipulation program FORM[28] to calculate colour factors for each diagram and bring the diagrams into a representation that explicitly contains information required at later steps in the calculation. For instance, this involves choosing automatically an optimal path (in most cases the shortest path) for the external momentum p to flow through each diagram (we are going to expand in p when the operator $\mathcal{P}_N$ is applied) and determining automatically the diagram's topology when p is nullified. This information, for all diagrams together, is kept in a single file and is accessible using MINOS. We instruct MINOS to run sequentially, one diagram at a time, a highly optimized FORM program that performs the explicit calculation i.e. it substitutes all the Feynman rules, it performs projections on the Lorentz structures of the Green functions, it Taylor-expands the diagram in the external momentum p (the depth of the expansion increases with the moment index N), it takes all the Dirac traces, contracts with the tensors $q^{\{\mu_1} \cdots q^{\mu_N\}}$ and finally calls the MINCER [29] integration package to perform the 3-loop scalar integrals of the massless propagator type (using the integration by parts algorithms published in Ref. [30]). The results together with some useful technical information about the calculation (such as the resources used) are again stored into a single file. MINOS will initiate the calculation of a next diagram as soon as the calculation of a previous diagram is completed without any need for human interference.

The complete set of the results of our calculations can be found in Ref. [13] where further details and an analysis of the results are given. The obtained 3-loop singlet anomalous dimensions and coefficient functions required the calculation of more than 10.000 3-loop diagrams. The calculation of these diagrams (for N=2,4,6,8) required more than the equivalent of 7500 hours on a 150 Mhz SGI Challenge workstation and at some instances 2 Gbyte of storage place for the intermediate stages in the calculation.

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