Errors in the Extraction of α_s from the Bjorken Sum Rule

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Abstract: We discuss the contributions of experimental and theoretical errors to the determination of α_s using data on the Bjorken sum rule at different values of Q^2. The experimental contribution to the error in α_s generally decreases as Q^2 decreases, whereas the theoretical errors increase. Padé summation reduces the theoretical contribution to the error to such an extent that the optimal value of Q^2 for measuring α_s may be quite low.

1 Introduction

We have described previously [1, 2] a method of extracting α_s from the Bjorken sum rule which uses Padé Summation (PS) to estimate the QCD perturbation series.[†] In this note, we analyze the error in $\alpha_s(M_Z)$ that may be obtained from the Bjorken sum rule, as a function of the experimental error and various sources of theoretical error, including the summation prescription, renormalization scale and scheme dependence, and higher-twist effects. Our prime aim is to estimate the dependence of combined theoretical and experimental errors in $\alpha_s(M_Z)$ on the momentum transfer Q^2 at which the analysis is made. This information may be of use in the planning of future polarized electron-nucleon scattering experiments, as it may enable the beam energy to be optimized for the extraction of $\alpha_s(M_Z)$.

2 Propagation of the Experimental Error

The way in which a given experimental error in measuring the Bjorken integral $\Gamma_1^p - \Gamma_1^n$ propagates into a derived error in α_s depends on Q^2. In experiments done at high Q^2 (Q_h^2) where α_s is small, the error in $\alpha_s(Q_h^2)$ is given directly by the error in the measured value of $\Gamma_1^p - \Gamma_1^n$:

$$\Delta(\Gamma_1^p - \Gamma_1^n)(Q_h^2) = \frac{1}{6} g_A \Delta f(x) \sim \frac{g_A}{6\pi} \Delta\alpha_s(Q_h^2) \tag{1}$$

[†] For a more complete list of references, see [3].

since the QCD correction factor $f(x) \sim 1 - x$ for small $x \equiv \alpha_s/\pi$. On the other hand, for experiments done at low Q^2 (Q_l^2) where α_s is large,

$$\Delta(\Gamma_1^p - \Gamma_1^n)(Q_l^2) = \frac{1}{6} g_A \Delta f(x) = \frac{g_A}{6\pi}(1 + Corr(x))\Delta\alpha_s(Q_l^2) \tag{2}$$

where, in $\overline{\rm MS}$ with $\mu = Q$, $Corr(x)$ is a *positive* correction function that results from the higher-order terms in the perturbative series for the Bjorken sum rule (see, for example, Fig. 2 in [1]), so that

$$\Delta\alpha_s(Q_l^2) \sim \frac{6\pi}{g_A}\left(\frac{1}{1 + Corr(x)}\right)\Delta(\Gamma_1^p - \Gamma_1^n)(Q_l^2) \tag{3}$$

Thus, in an experiment done at a lower momentum transfer, the same experimental error in $\Gamma_1^p - \Gamma_1^n$ would propagate to a smaller error in α_s. Although this is the most important effect on the comparison between the errors in α_s obtainable from experiments at different Q^2, it is not the whole story, since they must be evolved to the same reference value of Q^2, e.g., M_Z^2. As is well known, this evolution renders large uncertainties in α_s at low Q^2 equivalent to smaller uncertainties in α_s at higher Q^2, reinforcing the previous effect.

To evaluate these effects in more detail, we recall that to leading non-trivial order in the $\overline{\rm MS}$ renormalization scheme:

$$\Delta(\Gamma_1^p - \Gamma_1^n)(Q_l^2) = \frac{g_A}{6\pi}\frac{d}{dx}\left(x + (\frac{23}{12}C_a - \frac{7}{8}C_f - \frac{1}{3}N_f)x^2\right)\Delta\alpha_s$$

Specializing to three colours, we obtain

$$\Delta(\Gamma_1^p - \Gamma_1^n)(Q_l^2) = \frac{g_A}{6\pi}\left(1 + \frac{2}{\pi}(4.5833 - \frac{1}{3}N_f)\alpha_s(Q_l^2)\right)\Delta\alpha_s \tag{4}$$

which we may invert to yield

$$\Delta\alpha_s(Q_l^2) \sim \frac{6\pi}{g_A}\left(\frac{1}{1 + \frac{2}{\pi}\left(4.5833 - \frac{1}{3}N_f\right)\alpha_s(Q_l^2)}\right)\Delta(\Gamma_1^p - \Gamma_1^n)(Q_l^2) \tag{5}$$

In order to compare the error in an experiment at Q_l^2 to that in an experiment at Q_h^2, we use renormalization-group evolution to evaluate the error in $\alpha_s(Q_h^2)$ obtained from an experiment at Q_l^2. At leading order, we have:

$$\alpha_s(Q_h^2) = \alpha_s(Q_l^2) - \frac{\pi}{4}\left(11 - \frac{2}{3}N_f\right)t\frac{\alpha_s^2(Q_l^2)}{\pi^2} \tag{6}$$

where $t = ln\left(\frac{Q_h^2}{Q_l^2}\right)$ is positive, and therefore

$$\Delta\alpha_s(Q_h^2) = \left(1 - \frac{1}{2\pi}\left(11 - \frac{2}{3}N_f\right)t\alpha_s(Q_l^2)\right)\Delta\alpha_s(Q_l^2) \tag{7}$$

Thus, an experiment at low momentum transfer Q_l^2 yields an uncertainty

$$\Delta\alpha_s(Q_h^2) \sim \frac{6\pi}{g_A}\left(\frac{1 - \frac{1}{2\pi}\left(11 - \frac{2}{3}N_f\right)t\alpha_s(Q_l^2)}{1 + \frac{2}{\pi}\left(4.5833 - \frac{1}{3}N_f\right)\alpha_s(Q_l^2)}\right)\Delta(\Gamma_1^p - \Gamma_1^n)(Q_l^2) \tag{8}$$

This demonstrates explicitly the two reasons why the slope of $\Delta\alpha_s$ as a function of $\Delta(\Gamma_1^p - \Gamma_1^n)$ decreases with decreasing Q^2: the positivity of the higher-order perturbative corrections to the Bjorken sum rule makes the denominator larger than 1, and the running of α_s makes the numerator smaller than 1.

3 Theoretical Errors

The above discussion ignored uncertainties in the theoretical formula used to interpret the data, and we now turn to discuss how the various theoretical errors due to these uncertainties contribute to the error in the extraction of α_s from the Bjorken sum rule. The main source of uncertainty is simply that the perturbative QCD series for the Bjorken sum rule is known only up to next-to-next-to-leading-order in α_s. Moreover, this series is expected to be asymptotic, so the higher-order terms are significant. In [1, 2] we examined the utility of Padé Summation (PS) methods for the resummation of the series. We showed that using PS to extract α_s yields a result that is much less scale [1] and scheme [2] dependent than if the naive partial sum is used.

One method of evaluating the theoretical uncertainty in α_s due to higher-order terms in the series is to compare different summation prescriptions. In [1], we estimated this from the difference between the values of α_s extracted using the [2/2] PS (our preferred value) and using the [2/1] and [1/2] PS's.

Another method of evaluation is to consider the scale dependence of the value of α_s that is extracted. As we have demonstrated in [1], this is reduced significantly by using the PS method. We estimate the scale-dependence error from the differences between the values of α_s that we find when setting the scale μ^2 in the [2/2] PS of the $\overline{\rm MS}$ series to be Q^2 (our preferred value), and when we set the scale of the same function to be $(2Q)^2$ or $(\frac{1}{2}Q)^2$, taking for definiteness the average of these two. A similar result is obtained when one considers the two-parameter renormalization-scheme dependence of the PS [2].

A third theoretical uncertainty is in the size of the higher-twist effect, whose contribution to the Bjorken sum rule is currently estimated to be

$$-\frac{0.02}{Q^2} \pm \frac{0.01}{Q^2} \tag{9}$$

We note in passing that this magnitude is similar [1] to the estimated ambiguity in the Borel resummation of the perturbative QCD series.

All these three types of theoretical uncertainty are reduced at higher momentum transfer, for the simple reason that they are all associated with the unknown higher-order corrections, which become smaller at higher Q^2.

4 Combined Error

We see that the propagated experimental error in α_s tends to increase with increasing momentum transfer, while the theoretical errors have the opposite effect - they tend to decrease with

increasing momentum transfer. The interplay between these different behaviours is such that one must examine the combined experimental and theoretical error as a function of both the experimental error in $\Gamma_1^p - \Gamma_1^n$ and the momentum transfer.

We define the total theoretical error as the sum in quadrature of the three theoretical errors defined above, namely the prescription error, the scale- and scheme-dependence error, and the higher-twist error. This cannot be exactly correct, but it is the best we can offer, until the correlations between the different errors become clearer. Finally, we define the combined error as the sum in quadrature of the total theoretical error and the experimental error. This treatment may also be questioned, but again we do not have a better procedure.

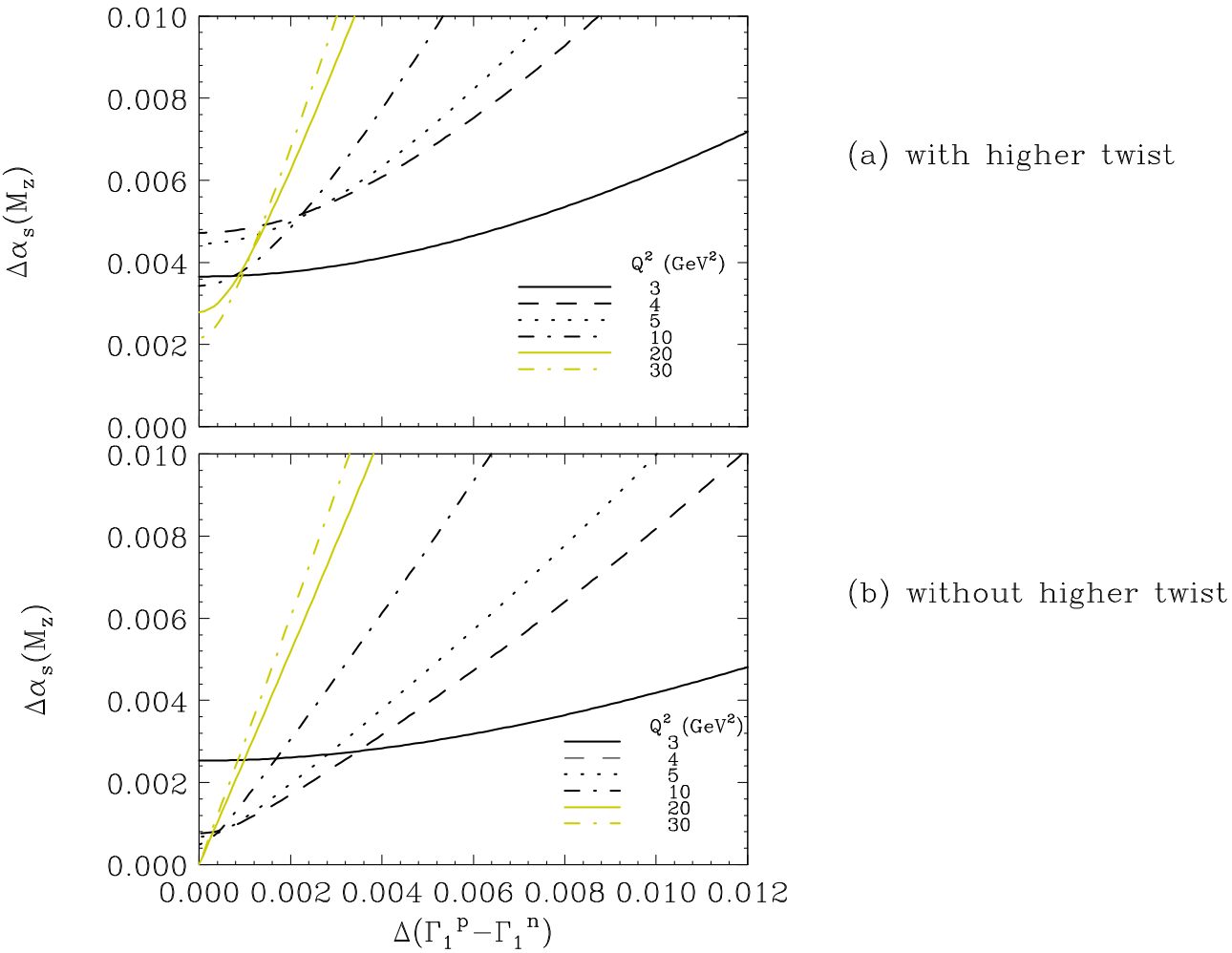

Figure 1: *(a) The error in $\alpha_s(M_Z)$ as a function of the experimental error in $\Delta[\Gamma_1^{(p-n)}(Q^2)]$, for $Q^2 = 3, 4, 5, 10, 30$ GeV^2. (b) The same as (a), but without the higher-twist correction.*

We now present a figure that shows the dependence of the combined error in $\alpha_s(M_Z)$ on the measurement error $\Delta(\Gamma_1^p - \Gamma_1^n)$. Fig. 1(a) shows the combined error as a function of the experimental error. We see that the slopes of the different Q^2 lines become steeper at higher Q^2.

This is due to the propagation of the experimental error described in Section 2. The values of $\Delta\alpha_s$ at which the lines for different Q^2 cross the $\Delta(\Gamma_1^p - \Gamma_1^n) = 0$ axis are the theoretical errors. We see the expected general trend that at higher Q^2 the theoretical error is lower. However, we also note that this general rule does not hold when there is a change in the number of contributing quark flavors, specifically as we go from $Q^2 = 3$ GeV^2 to $Q^2 = 4$ GeV^2.

In Fig. 1(b), both the magnitude and the error in the higher-twist effect are set to zero. For two reasons, this leads to a much smaller combined error. In the absence of the higher-twist shift, the central value of α_s and hence its experimental error is reduced, and the absence of the higher-twist error reduces the theoretical error. We see that, if there is no higher-twist effect, the higher Q^2 lines intersect the $Q^2 = 3$ GeV^2 line at much higher values of $\Delta(\Gamma_1^p - \Gamma_1^n)$.

5 Conclusions

This analysis indicates that, for a given error in $\Gamma_1^p - \Gamma_1^n$, experiments at high momentum transfer Q^2 generally yield larger errors in $\alpha_s(M_Z)$ than experiments of similar precision at low momentum transfer. This conclusion is dependent on our claim to understand the theoretical errors at the stated level, and will be true as long as the measurement error in $\Delta(\Gamma_1^p - \Gamma_1^n)$ is not reduced, down to the level where the theoretical error lurks. Our analysis indicates that the higher-twist effect is crucial for determining the optimal momentum transfer for this particular measurement. However, we emphasize that other values of the momentum transfer may be better for other measurements.

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