Universität Hamburg
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FAKULTÄT
FÜR MATHEMATIK,
INFORMATIK UND
NATURWISSENSCHAFTEN

DESY.

Mid Term Talk

Gravitational Wave Measurements with SRF Cavities

Speaker: Robin Löwenberg

Universität Hamburg, DESY

Gudrid Moortgat-Pick, Krisztian Peters, Tom Krokotsch, Daniel Klein

Preface

- This talk is mainly based on:

  SEBASTIAN A.R. ELLIS – *Revisiting Gravitational Wave Detection in a SRF Cavity* (DESY-Talk, March 11, 2021)

  ASHER BERLIN ET AL., *Detecting High-Frequency Gravitational Waves with Microwave Cavities,* arXiv:2112.11465v1, 2021

  ASHER BERLIN ET AL., *Axion Dark Matter Detection by Superconducting Resonant Frequency Conversion,* arXiv:1912.11048v1, 2019

Detecting High-Frequency Gravitational Waves with Microwave Cavities

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(Dated: December 23, 2021)

Axion Dark Matter Detection by Superconducting Resonant Frequency Conversion

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We propose an approach to search for axion dark matter with a specially designed superconducting radio frequency cavity, targeting axions with masses $m_a \lesssim 10^{-6}$ eV. Our approach exploits axion-induced transitions between nearly degenerate resonant modes of frequency $\sim$ GHz. A scan over axion mass is achieved by varying the frequency splitting between the two modes. Compared to traditional approaches, this allows for parametrically enhanced signal power for axions lighter than a GHz. The projected sensitivity covers unexplored parameter space for QCD axion dark matter for 10^{-8} eV $\lesssim m_a \lesssim 10^{-6}$ eV and axion-like particle dark matter as light as $m_a \sim 10^{-14}$ eV.

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Revisiting Gravitational Wave Detection in an SRF Cavity

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Based on:
210x.xxxxx
A. Berlin, R. T. D'Agnolo, SARE

SRGW2021, March 11 2021

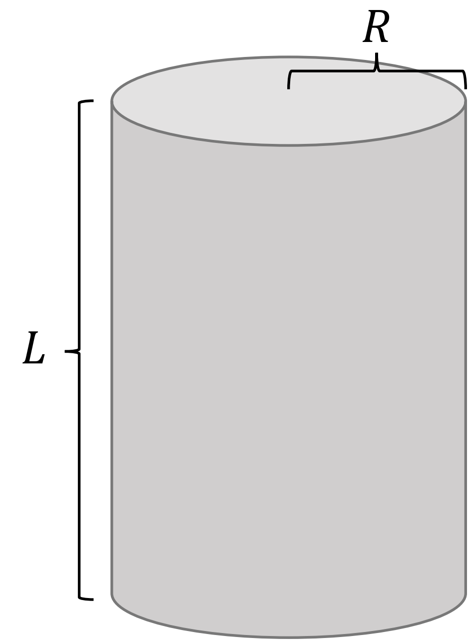
Outline

- Cavity Experiments – General Idea
- Coupling of Gravitational Waves to Cavity Modes
- High Frequency Gravitational Waves – Static B-Field Experiments
- Low Frequency Gravitational Waves – Heterodyne Experiments
    - Noise Sources

Cavity Experiments – Basics

- Gravitational Waves can couple to the EM-Field of a Cavity

  $\Rightarrow$ **We mainly focus on Cylindrical Cavities in this talk!**



- Two different types of modes:

  $\mathrm{TE}_{mnp\pm}$ $\Rightarrow$ Transverse Electric Mode ($E_z = 0$)

  $\mathrm{TM}_{mnp\pm}$ $\Rightarrow$ Transverse Magnetic Mode ($B_z = 0$)

- Frequency of lowest lying modes for $L \sim 1\mathrm{m}$

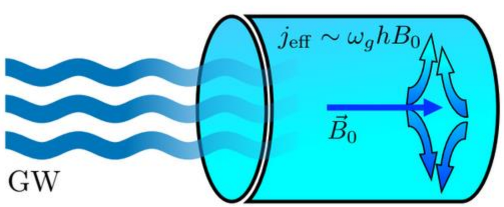
  $f \sim 300\ \mathrm{MHz}$

Static B-Field Experiments

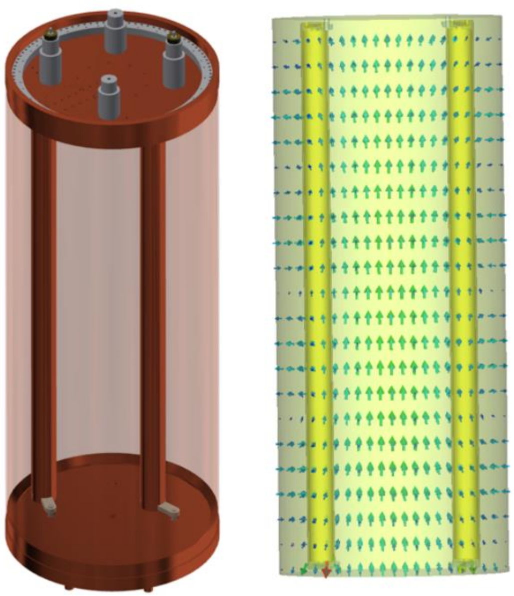
- Gravitational Waves is on-resonant with an **eigenmode of the cavity** and couples to a **static B-field**

- Good method to detect **High Frequency GW**
  $f \sim \mathcal{O}(\mathrm{GHz})$

- Method already well established in axion experiments (e.g. ADMX)



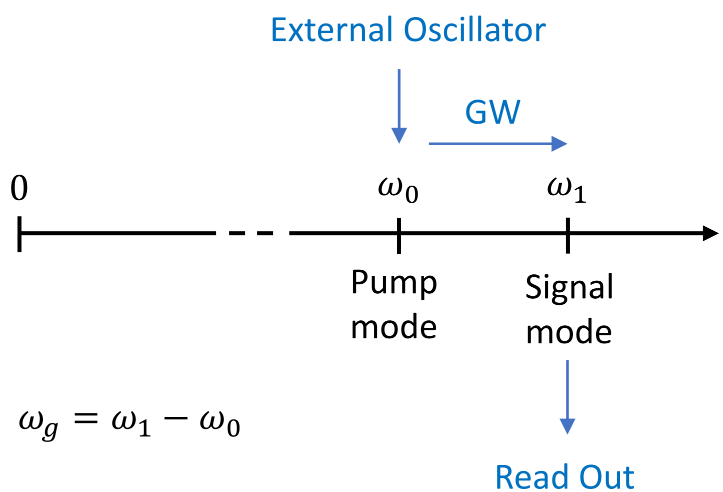
ASHER BERLIN ET AL., *Detecting High-Frequency Gravitational Waves with Microwave Cavities,* arXiv:2112.11465v1, 2021



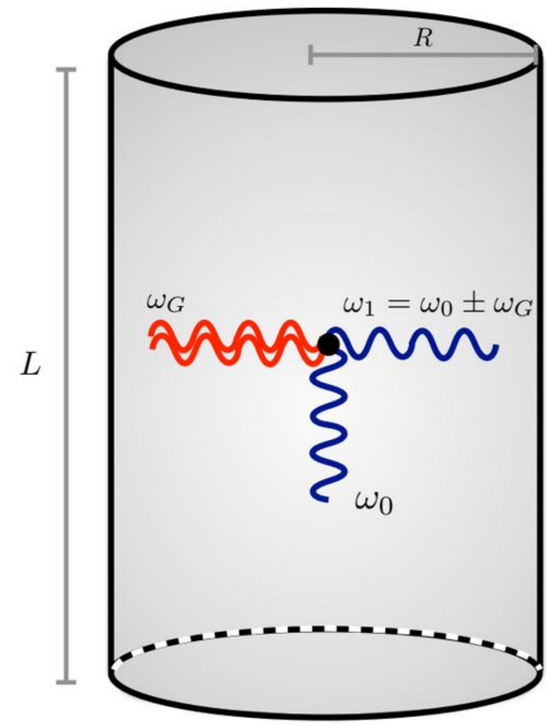
R. KHATIWADA ET AL., *Axion Dark Matter eXperiment: Detailed Design and Operations,* arXiv:2010.00169v1, 2020

Heterodyne Experiments

- GW is on-resonant with **the frequency difference** of two cavity modes and couples to both E- and B-Field



$$\omega_g = \omega_1 - \omega_0$$



SEBASTIAN A.R. ELLIS – *Revisiting Gravitational Wave Detection in a SRF Cavity* (DESY-Talk, March 11, 2021)

- Approach useful for low frequency GW measurements

$$f \sim \mathcal{O}(\mathrm{Hz} - \mathrm{MHz})$$

SRF Cavities

- Important quantity: **Quality Factor**
- Describes decay rate of excited mode m in cavity:

$$\frac{dP_m}{dt} = -\frac{\omega_m}{Q_m} P_m$$

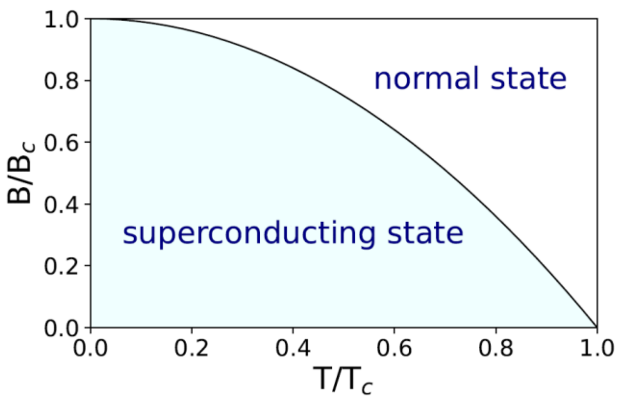
- High Quality Factors in **Superconducting Radio Frequency (SRF) Cavities**!

- Typical Parameters for Niobium Cavity:

$$T_c = 9.2 \text{ K} \qquad B_c = 0.2 \text{ T}$$

$$\Rightarrow \quad Q \sim \mathcal{O}(10^{10})$$

- We optimistically assume $Q \sim \mathcal{O}(10^{12})$



WERNER BUCKEL AND REINHOLD KLEINER. *Superconductivity.* John Wiley & Sons, Inc., 2004

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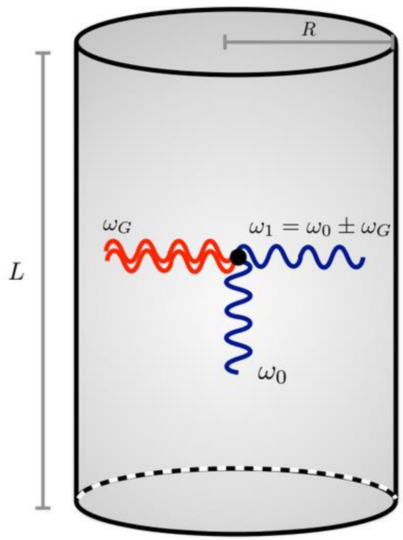
Possible Signals in a Cavity

GW can source two different signals in a cavity antenna

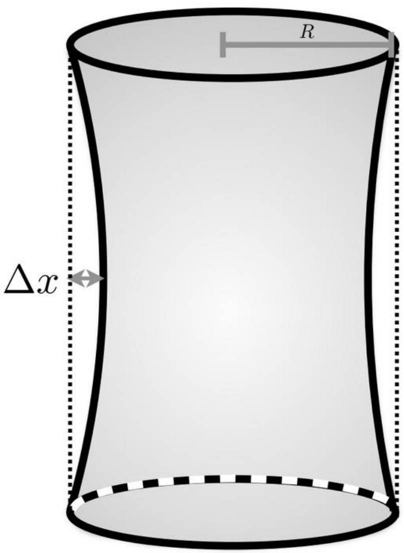
1.) Direct coupling to the EM-Field (Ghertsenshtein Effect) ⟵ **Focus**

2.) Deformation of the cavity walls (additional vibrational signal)

Ghertsenshtein-Effect



Wall Deformation



SEBASTIAN A.R. ELLIS – *Revisiting Gravitational Wave Detection in a SRF Cavity* (DESY-Talk, March 11, 2021)

Ghertsenshtein Effect

- Coupling of GW to electromagnetic field can be described in weak field limit by the Lagrangian

$$\mathcal{L}_{\mathrm{GZ}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} j^{\mu}_{\mathrm{eff}} A_{\mu}$$

- The effective current is

$$j^{\mu}_{\mathrm{eff}} = \partial_{\nu} \left(\frac{h^{\alpha}{}_{\alpha}}{2} F^{\mu\nu} + h^{\nu}{}_{\alpha} F^{\alpha\mu} - h^{\mu}{}_{\alpha} F^{\alpha\nu} \right)$$

Problem: This Current is not invariant under gauge transformation

- Equations of motion (in vacuum):

$$\nabla \cdot \vec{E} = j^{0}_{\mathrm{eff}}$$

$$\nabla \times \vec{B} - \partial_t \vec{E} = \vec{j}_{\mathrm{eff}}$$

GW Basics

- Gravitational Waves (GW) are described in the weak field approximation of general relativity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad\qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

- The GWs can be most easily described in TT-gauge

$$h^{0\mu} = 0 \qquad\qquad h_{\alpha}{}^{\alpha} = 0 \qquad\qquad \partial_i h^{ij} = 0$$

- E.g. a GW in z-direction takes the form

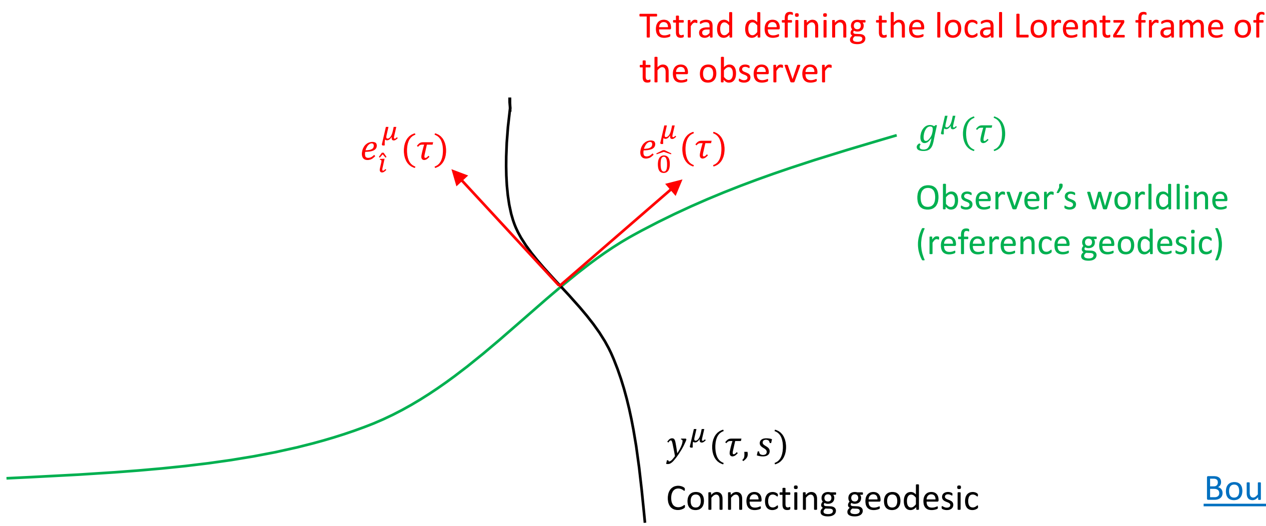
$$h_{\mu\nu} = A_{\mu\nu} e^{i\omega(t-z)} \qquad\qquad A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ **Problem: TT-Gauge is not a physical gauge!**

$\Rightarrow$ It does **not** describe a gravitational wave as seen by a local observer.

Fermi Normal Coordinates

- Better choice: **Fermi Normal Coordinates**!
- Construction:



Boundary Condition

$$\left.\frac{\mathrm{d}g^{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}y_{\mu}}{\mathrm{d}s}\right|_{s=0} = 0$$

- Geodesic equations:

$$\frac{\mathrm{d}^2 g^{\mu}}{\mathrm{d}\tau^2} + \Gamma^{\mu}_{\alpha\beta}\frac{\mathrm{d}g^{\alpha}}{\mathrm{d}\tau}\frac{\mathrm{d}g^{\beta}}{\mathrm{d}\tau} = 0 \qquad \frac{\mathrm{d}^2 y^{\mu}}{\mathrm{d}s^2} + \Gamma^{\mu}_{\alpha\beta}\frac{\mathrm{d}y^{\alpha}}{\mathrm{d}s}\frac{\mathrm{d}y^{\beta}}{\mathrm{d}s} = 0$$

Fermi Normal Coordinates

Most general metric in fermi normal coordinates (Local Lorentz Frame)!

$$g_{00} = -(1 + \vec{a} \cdot \vec{x})^2 + (\vec{\omega} \times \vec{x})^2 - \gamma_{00} - 2(\vec{\omega} \times \vec{x})^i \gamma_{0i} - (\vec{\omega} \times \vec{x})^i (\vec{\omega} \times \vec{x})^j \gamma_{ij}$$

$$g_{0i} = (\vec{\omega} \times \vec{x})_i - \gamma_{0i} - (\vec{\omega} \times \vec{x})^j \gamma_{ij}$$

$$g_{ij} = \delta_{ij} - \gamma_{ij}$$

The coefficients γ_{00}, γ_{0i} and γ_{ij} are given by

$$\gamma_{00} = \sum_{n=0}^{\infty} \frac{2}{(n+3)!} x^k x^l x^{k_1} \dots x^{k_n} \left(\partial_{k_1} \dots \partial_{k_n} R_{0k0l}\right)(g) \cdot [(n+3) + 2(n+2)\vec{a}\vec{x} + (n+1)(\vec{a}\vec{x})^2]$$

$$\gamma_{0i} = \sum_{n=0}^{\infty} \frac{2}{(n+3)!} x^k x^l x^{k_1} \dots x^{k_n} \left(\partial_{k_1} \dots \partial_{k_n} R_{0kil}\right)(g) \cdot [(n+2) + (n+1)\vec{a}\vec{x}]$$

$$\gamma_{ij} = \sum_{n=0}^{\infty} \frac{2}{(n+3)!} x^k x^l x^{k_1} \dots x^{k_n} \left(\partial_{k_1} \dots \partial_{k_n} R_{ikjl}\right)(g) \cdot [n+1]$$

KARL PETER MARZLIN. *“Fermi coordinates for weak gravitational fields”.* In: Physical Review D 50 (1994), pp. 888–891.

Simplifications

- The gravitational field of the earth is almost static and varies on typical frequencies $f \lesssim 0.1\ \mathrm{Hz}$.

- The effect of gravitational waves ($\mathcal{O}(\mathrm{kHz})$) can be well separated

- Metric with $\vec{a} = 0$ and $\vec{\omega} = 0$ simplifies to

$$h_{00} = -2 \sum_{n=0}^{\infty} \frac{n+3}{(n+3)!} x^k x^l x^{k_1} \dots x^{k_n} \left(\partial_{k_1} \dots \partial_{k_n} R_{0k0l}\right)(g)$$

$$h_{0i} = -2 \sum_{n=0}^{\infty} \frac{n+2}{(n+3)!} x^k x^l x^{k_1} \dots x^{k_n} \left(\partial_{k_1} \dots \partial_{k_n} R_{0kil}\right)(g)$$

$$h_{ij} = -2 \sum_{n=0}^{\infty} \frac{n+1}{(n+3)!} x^k x^l x^{k_1} \dots x^{k_n} \left(\partial_{k_1} \dots \partial_{k_n} R_{ikjl}\right)(g)$$

Better, but still complicated!

Simplifications

- Monochromatic gravitational wave along the z-axis:

$$\boxed{h_{\mu\nu}^{TT} = A_{\mu\nu} e^{i\omega_g(t-z)}}$$

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Riemann Tensor $R_{\mu\nu\alpha\beta}$ is **gauge invariant** and can be computed in TT-gauge.

- The metric further simplifies to:

$$h_{00} = -\omega_g h_{ab}^{TT} x^a x^b \left[-\frac{i}{\omega_g z} + \frac{1 - e^{-i\omega_g z}}{\left(\omega_g z\right)^2} \right]$$

$$h_{0i} = -\omega_g \left(h_{ia}^{TT} z x^a - \delta_{iz} h_{ab}^{TT} x^a x^b \right) \left[-\frac{i}{2\omega_g z} - \frac{e^{-i\omega_g z}}{\left(\omega_g z\right)^2} - i \frac{1 - e^{-i\omega_g z}}{\left(\omega_g z\right)^3} \right]$$

$$h_{ij} = \omega_g^2 \left(\left(\delta_{iz} h_{ja}^{TT} + \delta_{jz} h_{ia}^{TT} \right) z x^a - h_{ij}^{TT} z^2 - \delta_{iz} \delta_{jz} h_{ab}^{TT} x^a x^b \right) \left[-\frac{1 + e^{-i\omega_g z}}{\left(\omega_g z\right)^2} - 2i \frac{1 - e^{-i\omega_g z}}{\left(\omega_g z\right)^3} \right]$$

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High Frequency GW

- The gravitational wave induced current was given by

$$j^{\mu}_{\text{eff}} = \partial_{\nu}\left(\frac{h^{\alpha}{}_{\alpha}}{2}F^{\mu\nu} + h^{\nu}{}_{\alpha}F^{\alpha\mu} - h^{\mu}{}_{\alpha}F^{\alpha\nu}\right)$$

- We define a normalized current $\vec{j}_{+,\times}$ as

$$\vec{j}_{\text{eff}}(\vec{x}) \coloneqq B_0\omega_g^2 V_{\text{cav}}^{1/3}\left(h_+\vec{j}_+(\vec{x}) + \vec{h}_\times\vec{j}_\times(\vec{x})\right)$$

- A GW on resonant with a cavity mode $\vec{E}_n$ induces a signal

$$P_{\text{sig}} = \frac{1}{2}Q\omega_g^3 V_{\text{cav}}^{5/3}(\eta_n h_0 B_0)^2$$

$h_0 = h_+ \text{ or } h_0 = h_\times$

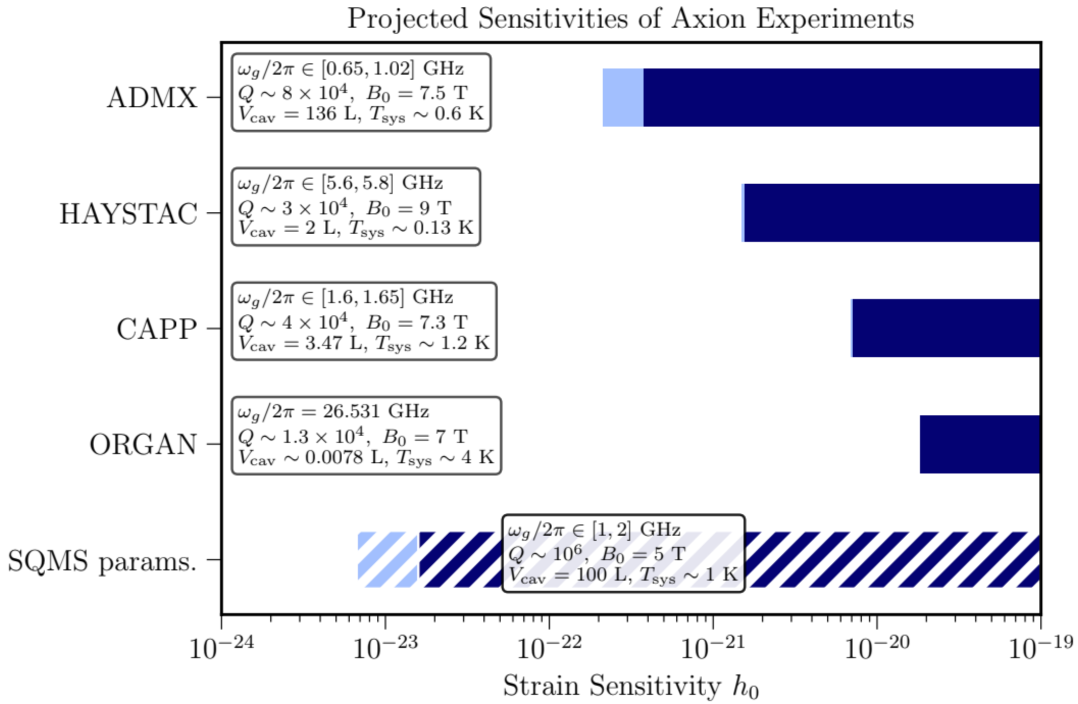
- Where the overlap factor η_n is defined by:

$$\eta_n \coloneqq \frac{\left|\int_{V_{\text{cav}}} d^3x\, \vec{E}_n^*(\vec{x}) \cdot \vec{j}_{+,\times}(\vec{x})\right|}{V_{\text{cav}}^{1/2}\left(\int_{V_{\text{cav}}} d^3x\, \left|\vec{E}_n(\vec{x})\right|^2\right)^{1/2}}$$

Asher Berlin et al. *Detecting High-Frequency Gravitational Waves with Microwave Cavities,* arXiv:2112.11465v1 (21.12.2021)

High Frequency Sensitivity

- Estimation of the sensitivity of current axion experiments: $P_{sig} = \frac{1}{2} Q \omega_g^3 V_{cav}^{5/3} (\eta_n h_0 B_0)^2$



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Long Wavelength Approximation

- Consider GW with long wavelengths ($\lambda_{\text{GW}} \gg 1\text{m}, \quad 1\text{m} \rightarrow 0.3\text{GHz}$)
- It is now sufficient to consider the metric expansion up to second order only
- Result:

$$\boxed{ds^2 = -dt^2\left(1 - \frac{1}{2}\ddot{h}_{kl}^{TT}(G)x^i x^j\right) + dx^i dx^j \delta_{ij}}$$

- So the only non vanishing component of the metric is:

$$h_{00} = \frac{1}{2}\ddot{h}_{kl}^{TT}(G)x^k x^l$$

Long Wavelength Approximation

- Remember the e.o.m and the current

$$\nabla \vec{E} = \rho_{\text{eff}}$$

$$\nabla \times \vec{B} - \partial_t \vec{E} = \vec{j}_{\text{eff}}$$

$$j^{\mu}_{\text{eff}} = \partial_{\nu} \left(\frac{h^{\alpha}{}_{\alpha}}{2} F^{\mu\nu} + h^{\nu}{}_{\alpha} F^{\alpha\mu} - h^{\mu}{}_{\alpha} F^{\alpha\nu} \right)$$

- Inserting h_{00} gives

$$\nabla E = \frac{1}{2} \nabla (h_{00} E_0)$$

$$\nabla \times B - \partial_t E = -\frac{1}{2} \underbrace{\left(\partial_t (h_{00} E_0) \right)}_{\text{Coupling to E-Field}} - \frac{1}{2} \underbrace{\left(\nabla \times (h_{00} B_0) \right)}_{\text{Coupling to B-Field}}$$

Power Spectral Densities

- The coupling to the E- and B-Field results into two independent Power Spectral Densities (PSD's) for the heterodyne setup

$$S_{\text{sig, E}}(\omega) = \frac{\omega_1}{4Q_1}(\eta_{10}^E E_0 H_0)^2 V \frac{\omega^4}{(\omega^2 - \omega_1^2)^2 + (\omega\omega_1/Q_1)^2} \int \frac{d\omega'}{2\pi} S_{h_0}(\omega - \omega') S_{e_0}(\omega')$$

$$S_{\text{sig, }B}(\omega) = \frac{\omega_1}{4Q_1}(\eta_{10}^B B_0 H_0)^2 V^{1/3} \frac{\omega^2}{(\omega^2 - \omega_1^2)^2 + (\omega\omega_1/Q_1)^2} \int \frac{d\omega'}{2\pi} S_{h_0}(\omega - \omega') S_{b_0}(\omega')$$

$$H_0 \coloneqq \sqrt{\frac{1}{V_{\text{cav}}} \int d^3x \, |h_{00}(\vec{x})|^2}$$

$$B_0 \coloneqq \sqrt{\frac{1}{V_{\text{cav}}} \int d^3x \, \left|\vec{B}_0(\vec{x})\right|^2}$$

$$E_0 \coloneqq \sqrt{\frac{1}{V_{\text{cav}}} \int d^3x \, \left|\vec{E}_0(\vec{x})\right|^2}$$

- Where the overlap factors η_{10}^E and η_{10}^B are given by

$$\eta_{10}^E \coloneqq \frac{\left|\int d^3\vec{x}\, \vec{E}_1^*(\vec{x}) \vec{E}_0(\vec{x}) h_{00}(\vec{x})\right|}{\sqrt{\int d^3\vec{x}\, \left|\vec{E}_1(\vec{x})\right|^2} \sqrt{\int d^3\vec{x}\, \left|\vec{E}_0(\vec{x})\right|^2} \sqrt{\frac{1}{V_{\text{cav}}} \int d^3\vec{x}\, |h_{00}(\vec{x})|^2}}$$

$$\eta_{10}^B \coloneqq \frac{V_{\text{cav}}^{1/3} \left|\int d^3\vec{x}\, \vec{E}_1^*(\vec{x}) \cdot \vec{\nabla} \times \left(\vec{B}_0(\vec{x}) h_{00}(\vec{x})\right)\right|}{\sqrt{\int d^3\vec{x}\, \left|\vec{E}_1(\vec{x})\right|^2} \sqrt{\int d^3\vec{x}\, \left|\vec{B}_0(\vec{x})\right|^2} \sqrt{\frac{1}{V_{\text{cav}}} \int d^3\vec{x}\, |h_{00}(\vec{x})|^2}}$$

Monochromatic GW

- Assuming Monochromatic GW, we can approximately write

$$S_h(\omega) = \pi^2 \left(\delta\left(\omega - \omega_g\right) + \delta\left(\omega + \omega_g\right)\right)$$

$$S_{e_0}(\omega) = S_{b_0}(\omega) = \pi^2\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

- Additionally assume GW in z-direction with certain polarization, such that (in SI-units)

$$h_+(\vec{x}) = \frac{1}{2}\frac{\omega_g^2}{c^2} h_+(y^2 - x^2) \qquad h_\times(\vec{x}) = -\frac{\omega_g^2}{c^2} h_\times xy$$

- This allows us to integrate over the Signal PSD. The Result is

$$\boxed{\begin{aligned} P_{\mathrm{sig,E}} &= \frac{1}{16\mu_0}\omega_1 Q_1 \pi \left(\eta_{10}^E \frac{E_0}{c} H_{+,\times}\right)^2 V_{\mathrm{cav}} \\ P_{\mathrm{sig,B}} &= \frac{1}{16\mu_0}\frac{Q_1}{\omega_1}\pi\left(\eta_{10}^B B_0 H_{+,\times}\right)^2 V_{\mathrm{cav}}^{1/3} \end{aligned}}$$

$$P_{\mathrm{sig}} = \frac{1}{(2\pi)^2}\int \mathrm{d}\omega \, S_{\mathrm{sig}}(\omega)$$

Overlap Factors

Recall:

$$\eta_{10}^{E} := \frac{\left|\int d^3\vec{x}\, \vec{E}_1^*(\vec{x})\vec{E}_0(\vec{x})h_{00}(\vec{x})\right|}{\sqrt{\int d^3\vec{x}\,\left|\vec{E}_1(\vec{x})\right|^2}\sqrt{\int d^3\vec{x}\,\left|\vec{E}_0(\vec{x})\right|^2}\sqrt{\frac{1}{V_{\mathrm{cav}}}\int d^3\vec{x}\,|h_{00}(\vec{x})|^2}} \qquad \eta_{10}^{B} := \frac{V_{\mathrm{cav}}^{1/3}\left|\int d^3\vec{x}\, \vec{E}_1^*(\vec{x})\cdot\vec{\nabla}\times\left(\vec{B}_0(\vec{x})h_{00}(\vec{x})\right)\right|}{\sqrt{\int d^3\vec{x}\,\left|\vec{E}_1(\vec{x})\right|^2}\sqrt{\int d^3\vec{x}\,\left|\vec{B}_0(\vec{x})\right|^2}\sqrt{\frac{1}{V_{\mathrm{cav}}}\int d^3\vec{x}\,|h_{00}(\vec{x})|^2}}$$

- Numerical Calculations for monochromatic GW in z-direction with polarization h_+

Modes	Radius	Length	ω_g	η_{10}^{E}	η_{10}^{B}
$\mathrm{TE}_{111} \rightarrow \mathrm{TE}_{121}$	2 m	0.02 m	5.97 MHz	0.17	17.97
$\mathrm{TM}_{121} \rightarrow \mathrm{TM}_{131}$	2 m	0.02 m	12.94 MHz	0.22	22.55

$\Rightarrow$ Use approximate values in the following: $\eta_{10}^{E} \approx 0.2$ $\eta_{10}^{B} \approx 20$

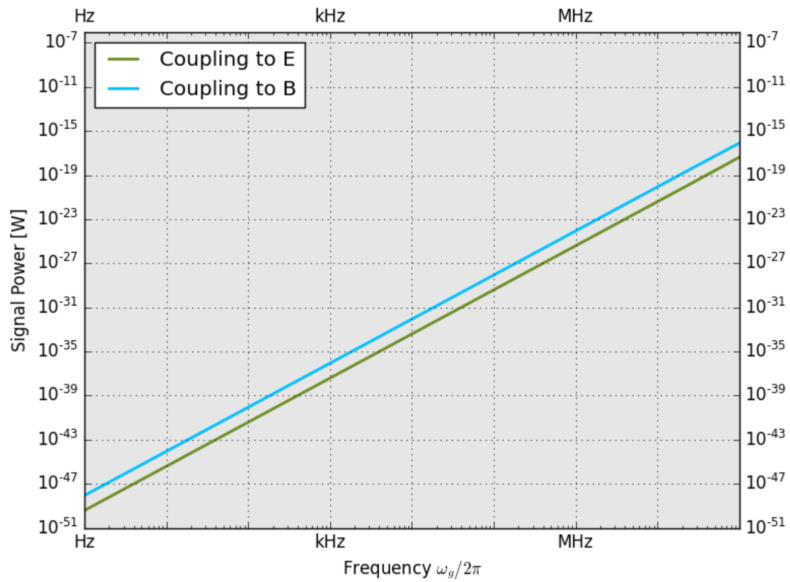
To-Do: Rigorous analysis of overlaps for more modes, cavity geometries and Gravitational Waves

Plot of the Signal

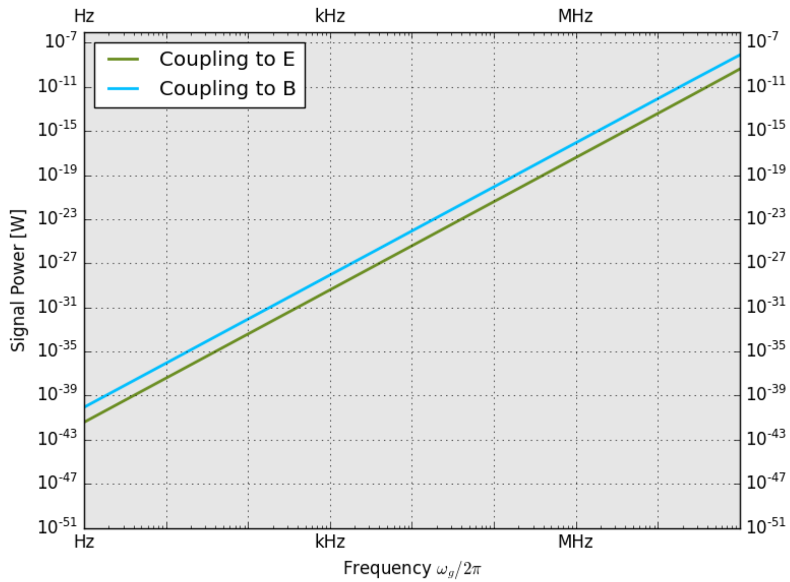
Parameters:

$$\eta_{10}^{B} = 20 \quad \eta_{10}^{E} = 0.2 \qquad B_0 = \frac{E_0}{c} = 0.2\ T \qquad H_{+,\times} = \frac{1}{2}\frac{\omega_g^2}{c^2} h_{+,\times} \cdot 0.001$$

$$\omega_0 = 2\pi \cdot 1\text{GHz} \qquad \omega_1 = \omega_0 + \omega_g \qquad Q_1 = 0.5 \cdot 10^{12}$$



$h_{+,\times} = 10^{-18}$



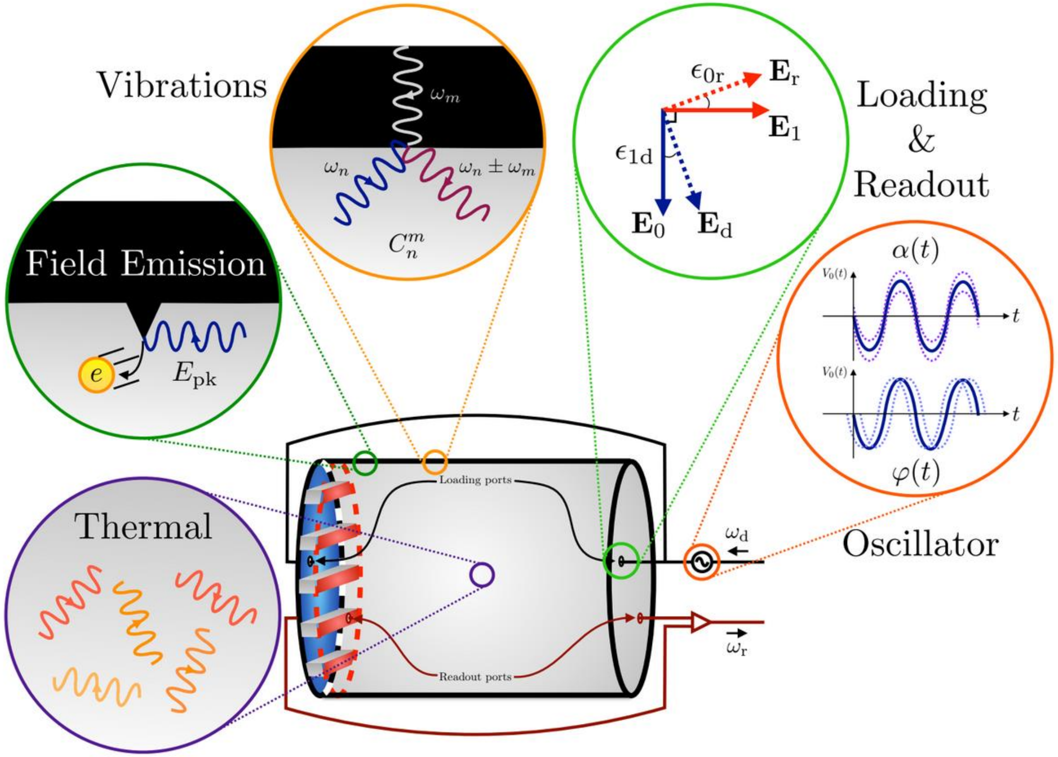
$h_{+,\times} = 10^{-14}$

Outline

- Cavity Experiments – General Idea
- Coupling of Gravitational Waves to Cavity Modes
- High Frequency Gravitational Waves – Static B-Field Experiments
- Low Frequency Gravitational Waves – Heterodyne Experiments
  - Noise Sources

Noice Sources

- Advantage of heterodyne setup: noise sources are already well investigated.



ASHER BERLIN ET AL., *Axion Dark Matter Detection by Superconducting Resonant Frequency Conversion,* arXiv:1912.11048v1, 2019

Noice Sources

- Every noise sources drives additional power into the signal mode which can be described by a PSD

- Thermal Noise (cavity walls):

$$S_{\mathrm{th}}(\omega) = \frac{Q_1}{Q_{\mathrm{int}}} \frac{4\pi T k_B (\omega\omega_1/Q_1)^2}{(\omega^2 - \omega_1^2)^2 + (\omega\omega_1/Q_1)^2}$$

- Amplifier Noise (amplifier of the readout system):

$$S_{\mathrm{ql}}(\omega) = 4\pi\hbar\omega_1$$

- Phase Noise (oscillator of the pump mode):

$$S_{\mathrm{phase}}(\omega) = \frac{1}{2}\epsilon_{1d}^2 S_\varphi(\omega - \omega_0) \frac{(\omega\omega_1/Q_1)^2}{(\omega^2 - \omega_1^2)^2 + (\omega\omega_1/Q_1)^2} \frac{\omega_0 Q_1}{\omega_1 Q_0} P_{\mathrm{in}}$$

Noise Sources

- Mechanical Noise (vibrations of the cavity eigenmodes)

$$S_{\mathrm{mech}}(\omega) = \frac{\epsilon_{1d}^2}{4}\frac{\omega_0}{Q_0}P_{\mathrm{in}}\sum_{n=0,1}\frac{1}{V^{2/3}}\frac{S_{q_m}(\omega-\omega_0)(\omega_n/Q_n)\omega_n^4\omega^2}{[(\omega^2-\omega_n^2)^2+(\omega\,\omega_n/Q_n)^2][(\omega_0^2-\omega_n^2)^2+(\omega_0\,\omega_n/Q_n)^2]}$$

With the displacement PSD

$$S_{q_m}(\omega) \approx \frac{1}{M^2}\frac{4\pi\omega_{\mathrm{min}}^3 q_{\mathrm{rms}}^2/Q_m}{(\omega^2-\omega_m^2)^2+(\omega_m\omega/Q_m)^2}$$

- All PSDs can be summed up

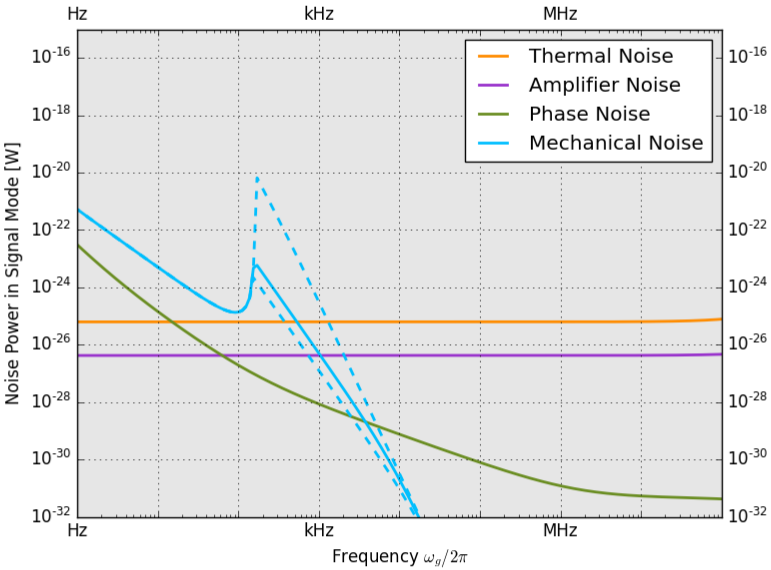
$$S_{\mathrm{noise}}(\omega) = S_{\mathrm{ql}}(\omega) + \frac{Q_1}{Q_{\mathrm{cpl}}}\left(S_{\mathrm{th}}(\omega) + S_{\mathrm{phase}}(\omega) + S_{\mathrm{mech}}^{(1)}(\omega)\right) + \frac{Q_0}{Q_{\mathrm{cpl}}}S_{\mathrm{mech}}^{(0)}(\omega)$$

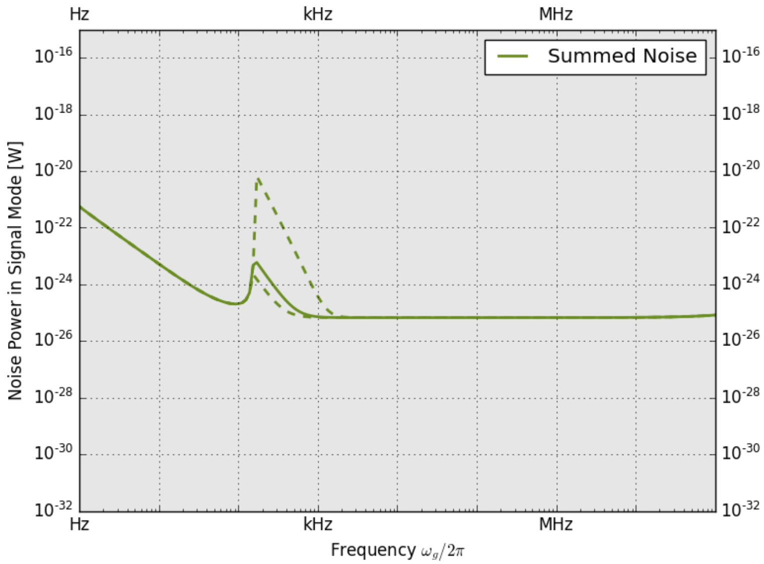
Plot of Noise Power

- We distinguish three different cases for the mechanical noise:

$$\omega_m = \min(\omega_{\min}, \omega_g) \qquad \omega_m = \min(\omega_{\min}, \omega_g + 25\text{ Hz}) \qquad \omega_m = \min(\omega_{\min}, \omega_g + 50\text{ Hz})$$

$$\omega_{\min} = 1\text{kHz}$$





$$\epsilon_{1d} = 10^{-7} \qquad Q_{\text{int}} = 10^{12}$$

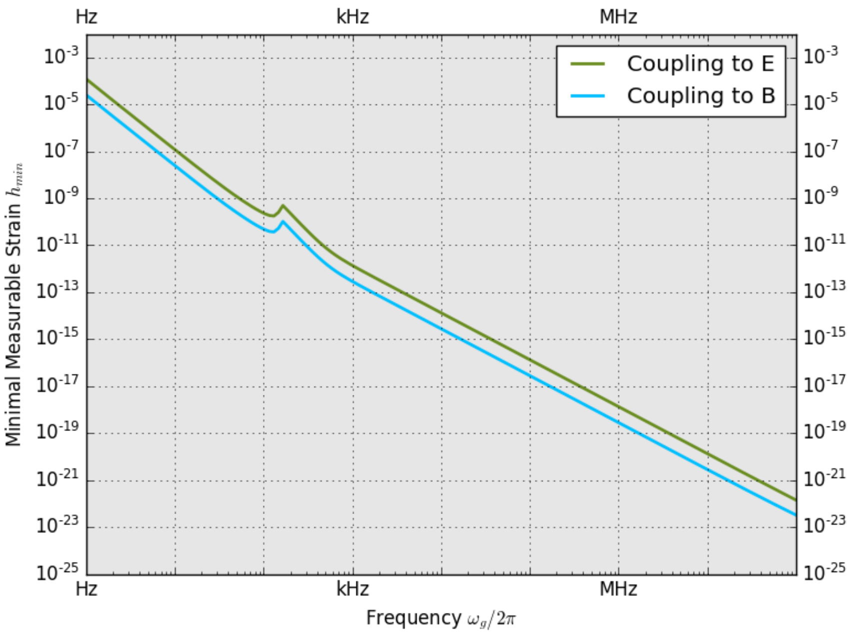
Parameters taken from:

ASHER BERLIN ET AL., *Axion Dark Matter Detection by Superconducting Resonant Frequency Conversion*, arXiv:1912.11048v1, 2019

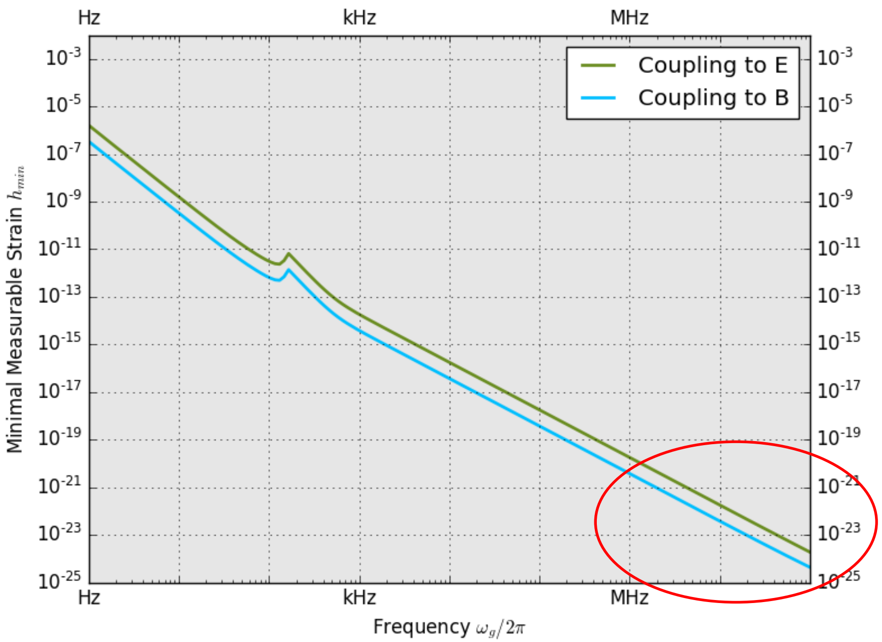
Minimal Measurable Strain

Estimation of the minimal strain we could measure for two different integration times τ

$$\mathrm{SNR} = \frac{P_{\mathrm{sig}}}{P_{\mathrm{noise}}}\sqrt{\tau} \overset{!}{=} 1 \quad \Rightarrow \quad h_{\min}$$



$\tau = 1s$



$\tau = 1a$

Conclusions

- Signal from Gertsenshtein-Effekt reaches reasonable high sensitivity at $\sim \mathcal{O}(\mathrm{kHz})$
- Predicted sources in that range:
    - Primordial Black Hole Mergers
    - Stochastic GW-Background from cosmic strings
    - Stochastic GW-Background from certain preheating models
    - Many more models …

N. Aggarwal et al. *Challenges and Opportunities of Gravitational Wave Searches at MHz to GHz Frequencies,* arXiv:2011.12414v2

To-Do

- Add analysis of signal from Stochastic GW Background

- Add more complex cavity geometries to the analysis

- Add the effect of wall deformation to the analysis

  ⇒ More sensitive in the low frequency range

  ⇒ Already well investigated by the MAGO collaboration

R. BALLANTINI ET AL. *Microwave Apparatus for Gravitational Waves Observation,* arXiv:gr-qc/0502054v1

To be continued …

Sources

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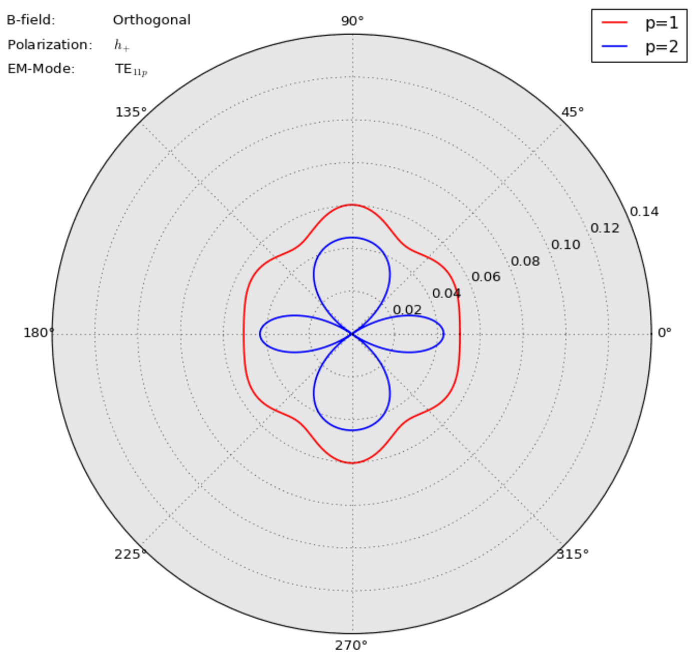
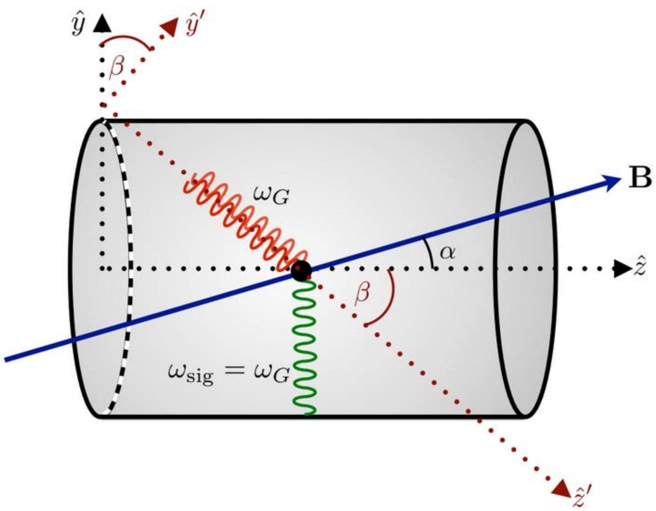
Backup Slides

High Frequency Overlap

- Overlap for GW from different directions in a cylindrical cavity with **static B-field**





ASHER BERLIN ET AL., *Detecting High-Frequency Gravitational Waves with Microwave Cavities,* arXiv:2112.11465v1, 2021

Parameters for the Noise Sources

Parameters:

- $B_0 = 0.2\ T$
- $q_{\mathrm{rms}} = 0.1 \cdot 10^{-9}\ \mathrm{m}$
- $Q_m = 1000$
- $\epsilon_{1d} = 10^{-5}$
- $\omega_0 = 10^9\ \mathrm{Hz}$
- $Q_0 = 10^{10}$
- $V = 1\ m^3$
- $\omega_1 = \omega_0 + m_a$
- $Q_1 = Q_0$
- $\omega_{\mathrm{min}} = 1000\ \mathrm{Hz}$

- ω_m nearest resonance, from 1 kHz to 10 kHz with 0.1 kHz steps

Axion-Photon Coupling

- Idea: The coupling of gravitational waves to the electromagnetic field is similar to the well known axion-photon coupling
- Effective Lagrangian:

$$\mathcal{L}_{\text{axion}-\text{photon}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu a \partial^\mu a - \frac{g_{a\gamma\gamma}}{4} a F_{\mu\nu} \tilde{F}^{\mu\nu}$$

- Equations of motion in vacuum and in the long wavelength regime:

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \times \vec{B} = \partial_t \vec{E} + g_{a\gamma\gamma} \vec{B}_0 \partial_t a =: \partial_t \vec{E} + \vec{J}_{\text{eff}}$$

$\Rightarrow$ **The axion drives an effective current into the cavity with B-field $\vec{B}_0$!**

Axion Power Spectral Densities

- The electromagnetic signal in a cavity is typically described by a Power Spectral Density (PSD)
- The signal power and PSD are related by: $P_{\text{sig}} = \frac{1}{(2\pi)^2} \int \mathrm{d}\omega \, S_{\text{sig}}(\omega)$
- Axion signal PSD for the signal mode $\vec{E}_1$ of a static B-field experiment (like ADMX):

$$S_{\text{sig}}(\omega) = \frac{\omega_1}{Q_1} \left(g_{a\gamma\gamma} \eta_{stat} B_0\right)^2 V_{\text{cav}} \frac{\omega^2}{(\omega^2 - \omega_1^2)^2 + (\omega\omega_1/Q_1)^2} S_a(\omega)$$

- Axion signal PSD for the pump mode $\vec{B}_0$ and signal mode $\vec{E}_1$ in a heterodyne setup:

$$S_{\text{sig}}(\omega) = \frac{\omega_1}{Q_1} \left(g_{a\gamma\gamma} \underbrace{\eta_{het}}_{} B_0\right)^2 V_{\text{cav}} \frac{\omega^2}{(\omega^2 - \omega_1^2)^2 + (\omega\omega_1/Q_1)^2} \int \frac{d\omega'}{(2\pi)^2} (\omega' - \omega)^2 S_{b_0}(\omega') S_a(\omega - \omega')$$

Important experimental quantity: overlap factor

Axion-Overlap-Factors

- Overlap factor for static B-field experiments:

$$\eta_{stat} = \frac{\left|\int_{V_{\mathrm{cav}}} \mathrm{d}^3x\, \vec{B}_0 \cdot \vec{E}_1^*(\vec{x})\right|}{B_0\sqrt{V \int_{V_{\mathrm{cav}}} \mathrm{d}^3x \left|\vec{E}_1(\vec{x})\right|^2}}$$

Typical Value for Cylindrical Cavities:
$\eta_{stat} = 0.69$

- Overlap factor for heterodyne experiments:

$$\eta_{het} = \frac{\left|\int_{V_{\mathrm{cav}}} \mathrm{d}^3x\, \vec{E}_1^* \cdot \vec{B}_0(\vec{x})\right|}{\sqrt{\int_{V_{\mathrm{cav}}} \mathrm{d}^3x \left|\vec{E}_1(\vec{x})\right|^2}\sqrt{\int_{V_{\mathrm{cav}}} \mathrm{d}^3x \left|\vec{B}_0(\vec{x})\right|^2}}$$

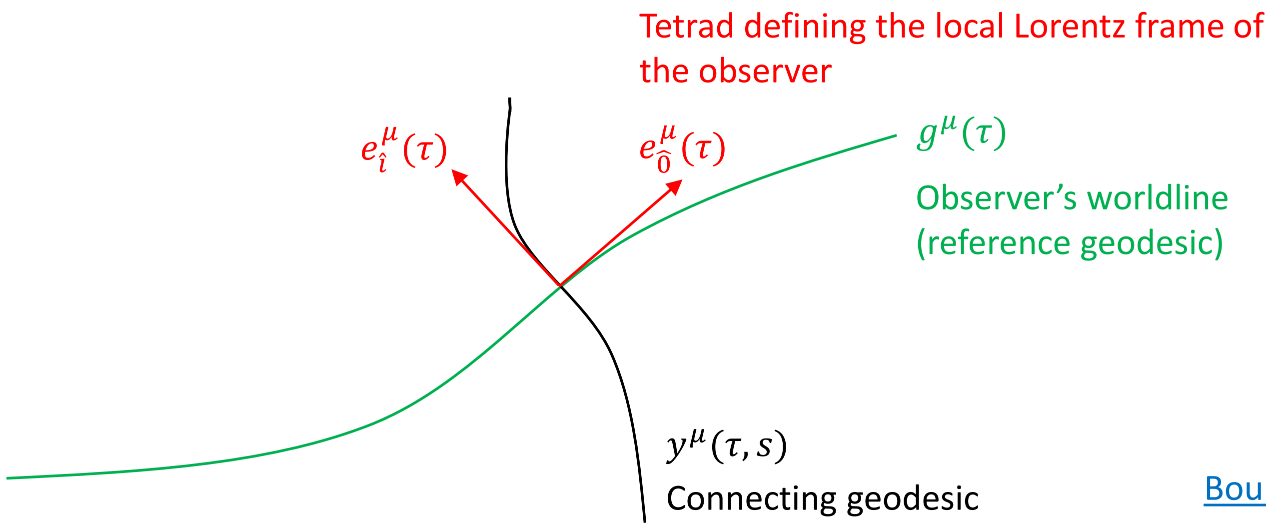
Typical Value for Cylindrical Cavities:
$\eta_{het} = 0.13$

⇒ Overlap factor strongly depends on the chosen modes and the cavity geometry

⇒ Goal: Choose experimental parameters such that $\eta \approx 1$

Fermi Normal Coordinates - Theory

- Better choice: **Fermi Normal Coordinates**!
- Construction:



Boundary Condition

$$\left.\frac{\mathrm{d}g^{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}y_{\mu}}{\mathrm{d}s}\right|_{s=0} = 0$$

- Geodesic equations:

$$\frac{\mathrm{d}^2 g^{\mu}}{\mathrm{d}\tau^2} + \Gamma^{\mu}_{\alpha\beta}\frac{\mathrm{d}g^{\alpha}}{\mathrm{d}\tau}\frac{\mathrm{d}g^{\beta}}{\mathrm{d}\tau} = 0 \qquad \frac{\mathrm{d}^2 y^{\mu}}{\mathrm{d}s^2} + \Gamma^{\mu}_{\alpha\beta}\frac{\mathrm{d}y^{\alpha}}{\mathrm{d}s}\frac{\mathrm{d}y^{\beta}}{\mathrm{d}s} = 0$$

Fermi Normal Coordinates - Theory

- In general, the tetrad can rotate:

$$\frac{\mathrm{d}e^{\mu}_{\hat{\alpha}}}{\mathrm{d}\tau} = -\Omega^{\mu}{}_{\nu} e^{\nu}_{\hat{\alpha}}$$

accalaration

$$\Omega^{\mu\nu} = a^{\mu}u^{\nu} - u^{\mu}a^{\nu} + u_{\alpha}\omega_{\beta}\epsilon^{\alpha\beta\mu\nu}$$

velocity

angular velocity

- Fermi normal coordinates are defined by: $v^{\mu} = x^{\hat{\alpha}} e^{\mu}_{\hat{\alpha}}$ $\quad x^{0} \coloneqq \tau$

- Solving the geodesic equation for y^{μ} gives:

$$y^{\mu}(x) = g^{\mu}(x^{0}) + x^{\hat{\imath}}\left(e^{\mu}_{\hat{\imath}}(x^{0}) + h_{\hat{\imath}}{}^{\mu}(x^{0})\right) - \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\partial_{\hat{k}_1} \dots \partial_{\hat{k}_n} h_{\hat{\imath}}{}^{\mu}\right)(g) x^{\hat{\imath}} x^{\hat{k}_1} \dots x^{\hat{k}_n}$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{(n+2)!}\left(\partial_{\hat{k}_1} \dots \partial_{\hat{k}_n} \partial^{\mu} h_{\hat{\imath}\hat{\jmath}}\right)(g) x^{\hat{\imath}} x^{\hat{\jmath}} x^{\hat{k}_1} \dots x^{\hat{k}_n}$$

- Remaining task: transform the metric with: $g_{\hat{\alpha}\hat{\beta}} = \frac{\partial y^{\mu}}{\partial x^{\hat{\alpha}}}\frac{\partial y^{\nu}}{\partial x^{\hat{\beta}}} g_{\mu\nu}(y(x))$

Fermi Normal Coordinates - Theory

Result: Very complicated metric!

$$g_{00} = -(1 + \vec{a} \cdot \vec{x})^2 + (\vec{\omega} \times \vec{x})^2 - \gamma_{00} - 2(\vec{\omega} \times \vec{x})^i \gamma_{0i} - (\vec{\omega} \times \vec{x})^i (\vec{\omega} \times \vec{x})^j \gamma_{ij}$$

$$g_{0i} = (\vec{\omega} \times \vec{x})_i - \gamma_{0i} - (\vec{\omega} \times \vec{x})^j \gamma_{ij}$$

$$g_{ij} = \delta_{ij} - \gamma_{ij}$$

The coefficients γ_{00}, γ_{0i} and γ_{ij} are given by

$$\gamma_{00} = \sum_{n=0}^{\infty} \frac{2}{(n+3)!} x^k x^l x^{k_1} \dots x^{k_n} \left(\partial_{k_1} \dots \partial_{k_n} R_{0k0l}\right)(g) \cdot [(n+3) + 2(n+2)\vec{a}\vec{x} + (n+1)(\vec{a}\vec{x})^2]$$

$$\gamma_{0i} = \sum_{n=0}^{\infty} \frac{2}{(n+3)!} x^k x^l x^{k_1} \dots x^{k_n} \left(\partial_{k_1} \dots \partial_{k_n} R_{0kil}\right)(g) \cdot [(n+2) + (n+1)\vec{a}\vec{x}]$$

$$\gamma_{ij} = \sum_{n=0}^{\infty} \frac{2}{(n+3)!} x^k x^l x^{k_1} \dots x^{k_n} \left(\partial_{k_1} \dots \partial_{k_n} R_{ikjl}\right)(g) \cdot [n+1]$$

Karl-Peter Marzlin. "Fermi coordinates for weak gravitational fields". In: Physical Review D 50 (1994), pp. 888–891.

Preparatory Project

Goal of the Preparatory Project

“By working on preparatory tasks, the student has acquired the special experimental and/or theoretical methods and knowledge of the field to such an extent that he or she can successfully apply them to work on issues from which the topic of the Master's thesis is to be derived”

Universität Hamburg, Table of Modules, M.Sc. Physics
https://www.physik.uni-hamburg.de/studium/dokumente/msc-physics-modultabelle-eng.pdf