Summary Master Thesis

The Catani-Seymour Dipole Subtraction Method with Massive Initial States in the Context of the Dark Matter Relic Abundance Calculation

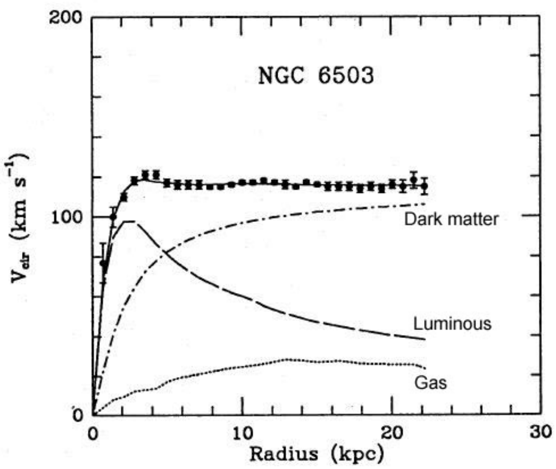
Mohamed Younes Sassi

Layout

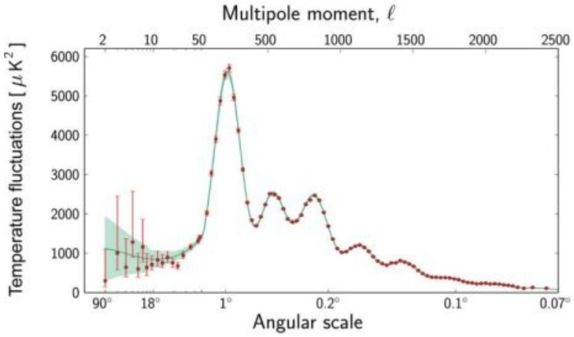
1. Dark matter evidence and relic abundance
2. IR divergences in QCD
3. Dipole subtraction method
4. Developing dipole subtraction method for the fully massive case
5. Implementation of the dipole method
6. Results and comparison with phase space slicing

Evidence for dark matter

There are lots of evidence for the existence of a non-relativistic non-baryonic matter in the universe which makes up ~ 26% of the energy of the universe



Rotation curves of galaxies



CMB temperature fluctuations

Assuming a thermal production of dark matter (DM) in the early universe and thermal freeze out, we can write a Boltzmann equation describing the evolution of dark matter density in the universe and deduce from it the present abundance of dark matter.

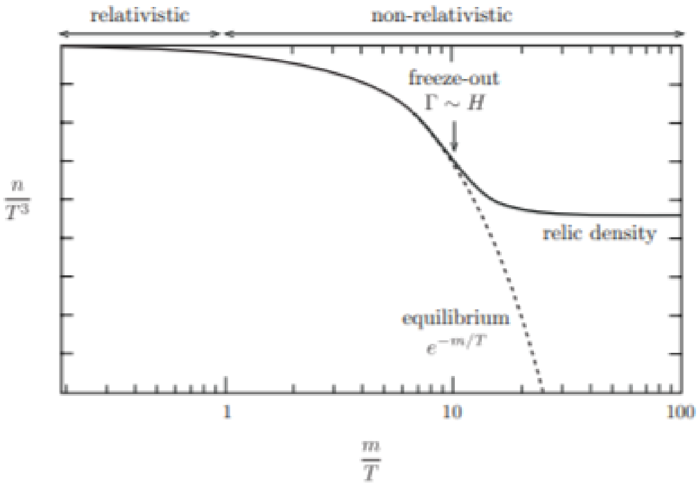
$$\dot{n}_\chi + 3Hn_\chi = -\langle \sigma_{eff} v_{rel} \rangle (n_\chi^2 - n_{\chi,eq}^2)$$

Dark matter relic density

- Experimental results for DM relic abundance very precise (Planck 2018)

$$\Omega_{DM}h^2 = 0.120 \pm 0.001$$

- Need to increase the precision in theoretical calculations of the Boltzmann equation.
- One possibility : **consider Next-to-leading order(NLO) corrections to σ_{eff}**



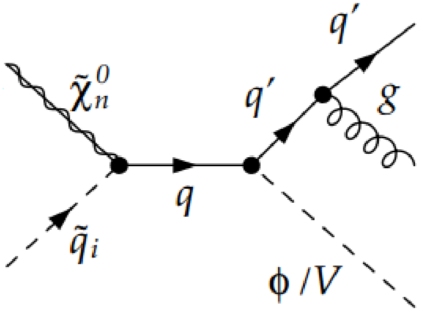
Source: Daniel Baumann TASI Lecture notes on primordial cosmology

$$\dot{n}_\chi + 3Hn_\chi = -\langle \sigma_{eff} v_{rel} \rangle (n_\chi^2 - n_{\chi,eq}^2)$$

Dark matter relic density

- NLO corrections include **real** and **virtual contributions.**

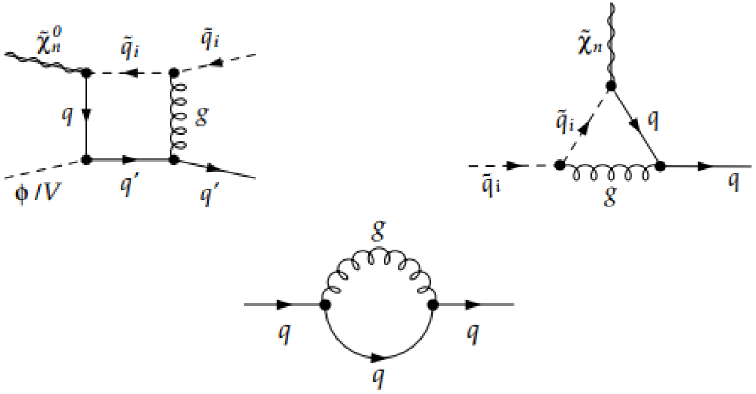
Real contributions:



2 to 3 processes

Involving emission of e.g: gluon (QCD), photon (QED),...

Virtual contributions:



2 to 2 processes

Involving loop calculations

Problem:

Virtual contributions induce **ultraviolet (UV)** and **infrared (IR)** divergences.

Real contributions induce **infrared** divergences.

Cancel ultraviolet divergences using renormalization techniques.

All UV divergences absorbed in **counterterms**.

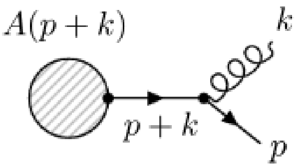
Infrared divergences cancel when considering the full NLO correction according to **Kinoshita-Lee-Nauenberg theorem**.

Add both real and virtual contributions:
IR divergences in real and virtual contributions cancel each other analytically.

$$\Longrightarrow \boxed{\int d\Phi_3 \;\; d\sigma_{real} + \int d\Phi_2 \;\; d\sigma_{virtual} = UV \;\; and \;\; IR \;\; finite}$$

IR Divergences in QCD

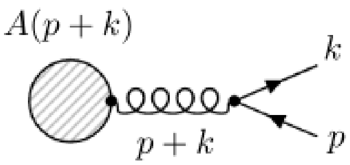
In general, infrared divergences can be described in 3 classes:
soft, **collinear** or **soft-collinear**



<u>“Soft” divergence</u>

Example: Massive quark emits a **soft** gluon.

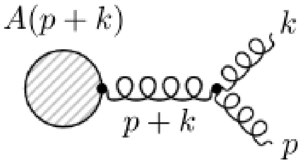
$$\frac{1}{(p+k)^2 - m^2} = \frac{1}{2p\cdot k} \xrightarrow{k\to 0} \infty$$



<u>“Collinear” divergence</u>

Example: Gluon decays into a **collinear massless quark antiquark pair**.

$$\frac{1}{(p+k)^2} = \frac{1}{\left|\vec{k}\right| \left(|\vec{p}| - \cos(\theta)\,|\vec{p}|\right)} \xrightarrow{\theta\to 0} \infty$$



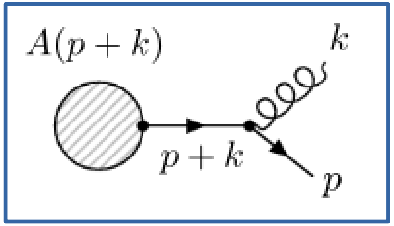
<u>“Soft”, “collinear” or “soft-collinear” divergence</u>

Example: Gluon emits a **soft** gluon or/and both gluons **parallel to each other**.

$$\frac{1}{(p+k)^2} = \frac{1}{|\vec{k}|(|\vec{p}| - \cos(\theta)|\vec{p}|)} \xrightarrow[and/or \quad \theta\to 0]{|\vec{k}|\to 0} \infty$$

Soft divergences

Consider the process of gluon emission off a final state quark.



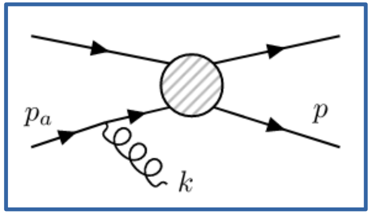
$$M_3 = \bar{u}(p)(igT^a\gamma^\mu)\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}A(p+k)\epsilon_\mu(k)$$

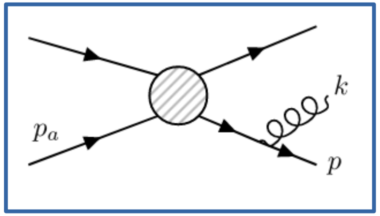
Now we use the eikonal approximation for the soft limit $k^\mu \to 0$

$$\Rightarrow M_3 = \bar{u}(p)(igT^a\gamma^\mu)\frac{im}{p\cdot k}A(p)\epsilon_\mu(k)$$

In case we have multiple particles in the final/initial state that can emit a gluon:

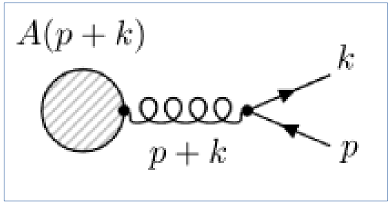
$$\Rightarrow |M_3|^2 = 8\pi\alpha_s C_F N_C \mu^{2\epsilon}\left(\frac{m^2}{(p\cdot k)^2}+\frac{m_a^2}{(p_a\cdot k)^2} - \frac{2p_a\cdot p}{(p_a\cdot k)(p\cdot k)}\right)|M_2|^2$$





Collinear divergence

Consider the process of decay of a gluon into a quark-antiquark pair.



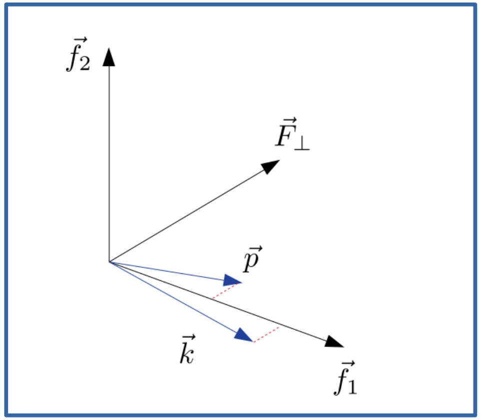
We work in d=4-2ε dimensions:

$$|M_3|^2 = 4\pi\alpha_s T_r \mu^{2\epsilon} A_0^\mu(p+k) \frac{\gamma_\mu \not{p} \gamma^\nu \not{k}}{(2p\cdot k)^2} A_{0,\nu}^*(p+k)$$

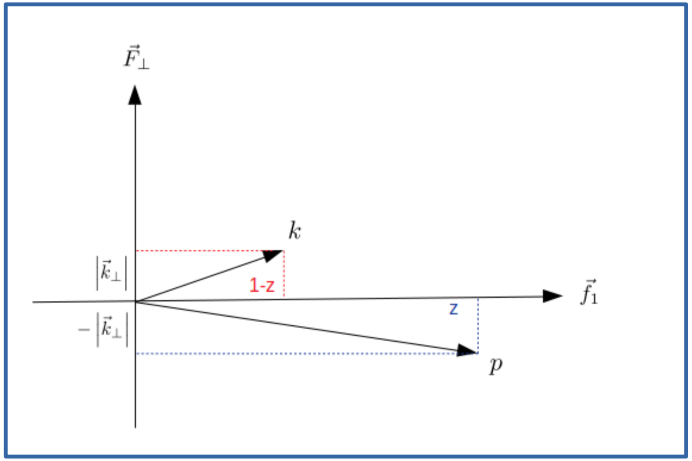
Define the Sudakov notation, which is useful to parameterize the collinear divergence of two momenta:

$$p^\mu = z f_1^\mu + k_\perp^\mu - \frac{k_\perp^2}{2z_1 \cdot f_2} f_2^\mu$$

$$k^\mu = (1-z) f_1^\mu - k_\perp^\mu - \frac{k_\perp^2}{2(z-1)_1 \cdot f - 2} f_2^\mu$$



Collinear divergence



Plug in the expression for p and k in the Sudakov parametrization in the matrix element and taking the collinear limit

$$|\vec{k}_\perp| \to 0 \Longrightarrow |M_3|^2 \xrightarrow{collinear} 8\pi\alpha_s \mu^{2\epsilon} T_r \langle M_{2,\mu}(f_1)| \frac{P^{\mu\nu}_{gq}}{2p\cdot k} |M_{2,\nu}(f_1)\rangle$$

P_{gq} is the Altarelli-Parisi splitting function.

$$P^{\mu\nu}_{gq} = (-g^{\mu\nu} - 4\frac{K^\mu_\perp K^\nu_\perp}{2p\cdot k})$$

In the case of a gluon:

$$P^{\mu\nu}_{gg} = 2C_a\left[-g^{\mu\nu}(\frac{z}{1-z} + \frac{1-z}{z}) - (2-2\epsilon)\frac{K^\mu_\perp K^\nu_\perp}{2p\cdot k}\right]$$

Summarizing the possible IR divergences

We can write down the behavior of the matrix element squared as:

$$|M_3|^2 \rightarrow \sum_i \sum_{i \neq j} \frac{1}{S_i} \langle M_2 | V_{i,j} | M_2 \rangle$$

$$S_i = (p_i + k)^2 - m_i^2$$

<u>Scalar propagator</u>

Soft limit: $V_{i,j} = 8\pi\alpha_s C_x N_c \mu^{2\epsilon} \left(\frac{m_i^2}{p_i \cdot k} \pm \frac{p_i \cdot p_j}{p_j \cdot k} \right)$

Collinear limit: $V_{i,j} = 8\pi\alpha_s N_c \mu^{2\epsilon} P_i$

We can analytically cancel all IR divergences when adding both virtual and real contributions.

Problem: Numerically, we calculate the contributions in two different phase spaces

$$\int d\Phi_3 \;\; d\sigma_{real} + \int d\Phi_2 \;\; d\sigma_{virtual}$$

Solution:

- Find a method to cancel the infrared divergences separately in both contributions.

  Already lots of methods available:

- **Phase space slicing** (Harris 2001)
- **Dipole subtraction methods** (Catani,Seymour 96, Dittmaier 99, Catani, Seymour 2002)

Dipole subtraction method

$$\sigma_{NLO} = \int d\Phi_3 \left[(d\sigma_{real})_{\epsilon=0} - (d\sigma_A)_{\epsilon=0} \right] + \int d\Phi_2 \left[d\sigma_{virtual} + \int d\Phi_{dipole} \;\; d\sigma_A \right]_{\epsilon=0}$$

Subtract an auxiliary cross section σ_A from the **real contribution** and add it back to the **virtual contribution**.

This auxiliary cross section should have the properties:

- Have the same divergent behavior in the soft/collinear limits as the real contribution.
- Must be easily analytically integrable over d-dimensional one-particle phase space of the gluon.

Dipole subtraction method in the literature

Dipole subtraction method initially developed for collider physics:

Massless particles (**Catani-Seymour 1996)**

Massive particles in the final state (**Catani-Seymour 2002)**

Problem with dark matter processes:

Initial state particles cannot be taken as massless, as it can also include squarks

Treatment of initial massive states done by (**Dittmaier 1999)** but only considered QED processes and mass regularization. Another work by (**Kotko 2012**) considered QCD processes but used another notation.

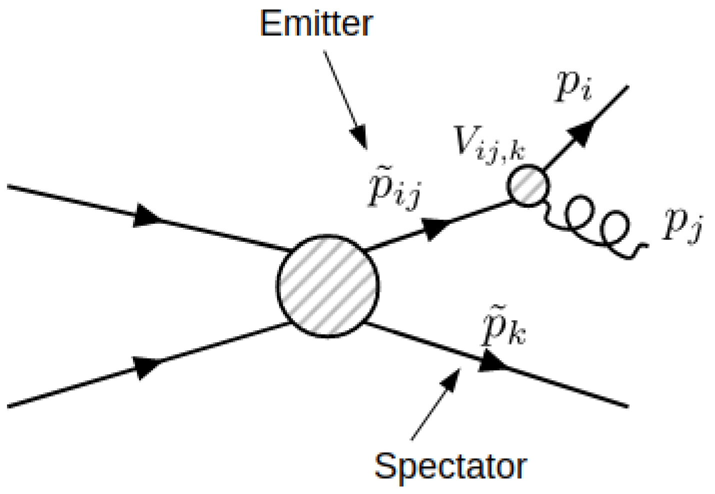
$\Longrightarrow$

We need to extend the formalism to also include massive initial states for QCD and using dimensional regularization.

General structure of the dipoles

$$d\sigma_A = \sum_{dipoles} D_{ij,k}$$

$$D_{ij,k} = \frac{1}{S_{ij}} \langle \tilde{p}_{ij}, \tilde{p}_k | \frac{T_{ij} T_k}{T_k^2} V_{ij,k} | \tilde{p}_{ij}, \tilde{p}_k \rangle$$



Ingredients:

1) S_{ij} usual scalar propagator of emitter.
2) $V_{ij,k}$ Dipole splitting functions. Have the same divergent behavior as $V_{i,j}$ in M_3 .
3) $|\tilde{p}_{ij}, \tilde{p}_k\rangle$ The tree matrix element written in terms of the dipole momenta.

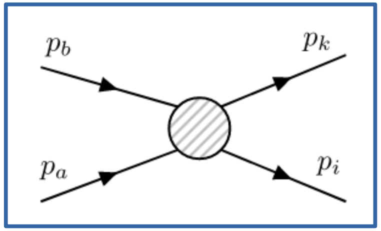
In contrast to the usual momenta, dipole momenta $\tilde{p}_k$ and $\tilde{p}_{ij}$ satisfy 2 to 2 momentum conservation $\Longrightarrow$ they will be used in the tree matrix element

How to choose the dipoles ?

Naively we can choose: $D_{ij,k} = \frac{1}{S_i}(V_{i,k}^{soft} + V_{i,k}^{collinear}) \otimes |M_2|^2$ same as $|M_3|^2_{soft/collinear} = \frac{1}{S_i} V_{i,k} \otimes |M_2|^2$

Problem 1
In the soft and collinear limit, we end up counting the soft limit two times because the Altarelli-Parisi splitting functions are also divergent in the soft limit.
Solution:
Choose a single splitting function for all limits.

Problem 2
We have momentum conservation for M_2 only at the strict soft limit. Outside this limit, M_2 is not defined as the Mandelstam variables are only defined for $p_a + p_b = p_i + p_k$
Solution:
Contruct new momenta that always respect momentum conservation and build the matrix element with them. $p_a + p_b = \tilde{p}_{ij} + \tilde{p}_k$



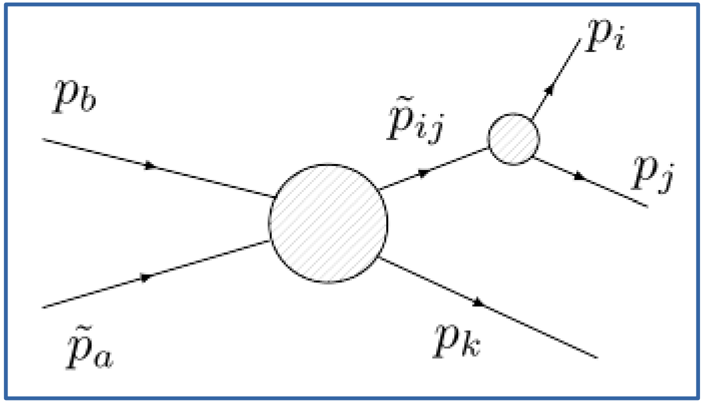
Constructing the dipoles

Final state emission, Initial state spectator $D^{FE\text{-}IS}$

Steps:

- Dipole kinematics
- Dipole splitting functions
- Phase space factorization
- Integration

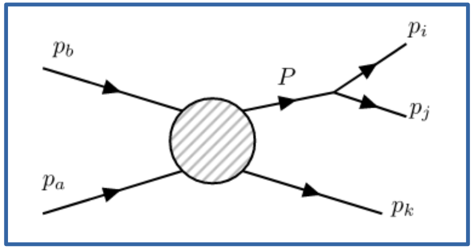
$$D_{ij,a}^{FE-IS}$$



“Feynman diagram” of the dipole function

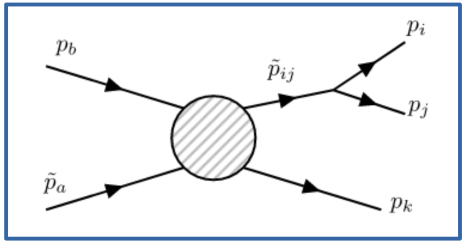
Dipole Kinematics

In order to get well defined tree matrix elements, we need to impose momentum conservation in all points of the phase space for 2 to 2 processes.



$$p_a + p_b = p_i + p_j + p_k$$

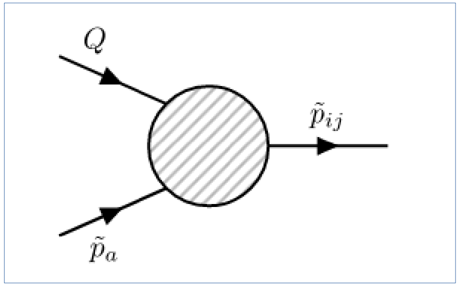
2 to 3 momentum conservation



$$\tilde{p}_a + p_b = \tilde{p}_{ij} + p_k$$

2 to 2 momentum conservation

$$Q^\mu = \tilde{p}^\mu_{ij} - \tilde{p}^\mu_a = p^\mu_b - p^\mu_k$$



Motivates writing the dipole momenta as a linear combination of Q and p_a:

$$\tilde{p}^\mu_{ij} = f(P^2, Q^2)Q^\mu + g(P^2, Q^2)p^\mu_a$$
$$\tilde{p}^\mu_a = \big(f(P^2, Q^2) - 1\big)Q^\mu + g(Q^2, P^2)p^\mu_a$$

Dipole momenta need to satisfy the conditions:

$$\tilde{p}_a + p_b = \tilde{p}_{ij} + p_k$$

Momentum conservation

$$\tilde{p}^2_{ij} = m^2_i, \;\; \tilde{p}^2_a = m^2_a$$

On-shell relations

$$\tilde{p}_{ij} \xrightarrow{soft} p_i$$
$$\tilde{p}_a \xrightarrow{soft} p_a$$

Soft limit

Dipole Kinematics

We end up with the system of equations:

$$m_i^2 = f^2 Q^2 + 2(Q \cdot p_a) f g + g^2 m_a^2$$
$$m_a^2 = (f-1)^2 Q^2 + 2(Q \cdot p_a)(f-1) g + g^2 m_a^2$$

Solve for f and g:

$$\tilde{p}_{ij}^\mu = \left(\frac{Q^2 - m_a^2 + m_i^2}{2Q^2} - \frac{\sqrt{\lambda_{ai}}}{\sqrt{\lambda(P^2, Q^2, m_a^2)}} \frac{Q \cdot p_a}{Q^2} \right) Q^\mu + \frac{\lambda_{ai}}{\sqrt{\lambda(P^2, Q^2, m_a^2)}} p_a^\mu$$
$$\tilde{p}_a^\mu = \left(\frac{Q^2 - m_a^2 + m_i^2}{2Q^2} - \frac{\sqrt{\lambda_{ai}}}{\sqrt{\lambda(P^2, Q^2, m_a^2)}} \frac{Q \cdot p_a}{Q^2} - 1 \right) Q^\mu + \frac{\lambda_{ai}}{\sqrt{\lambda(P^2, Q^2, m_a^2)}} p_a^\mu$$
$$\lambda_{ai} = \lambda(Q^2, m_a^2, m_i^2) = (Q^2 - (m_a^2 + m_i^2))^2 - 4 m_a^2 m_i^2$$

Dipole Splitting Functions

Define the dipole variables:

$$x_{ij,a} = \frac{p_a \cdot p_i + p_a \cdot p_j - p_i \cdot p_j}{p_a \cdot p_i + p_a \cdot p_j}$$

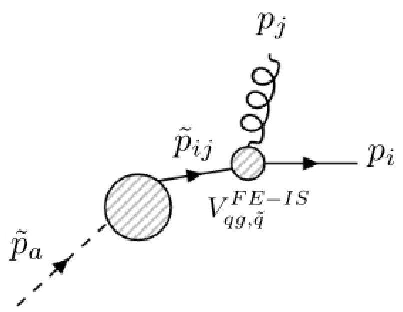
Parameterize the energy of the gluon.

$$z_i = \frac{p_a \cdot p_i}{p_a \cdot p_i + p_a \cdot p_j}$$

Describes angle between emitter and spectator

$$z_j = \frac{p_a \cdot p_j}{p_a \cdot p_i + p_a \cdot p_j} = 1 - z_i$$

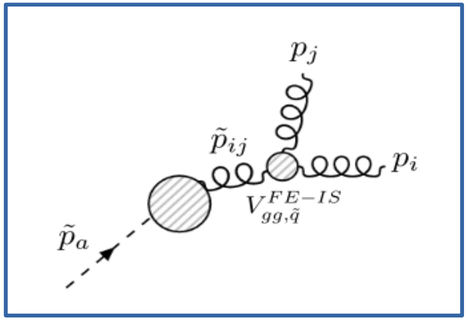
Describes the angle between the emitted particle and the spectator



Final state emission of a gluon off a quark, with the initial state squark as spectator

$$V_{qg,\tilde{q}}^{FE-IS} = 8\pi\alpha_s C_f N_c \mu^{2\epsilon}\left[\frac{2}{2 - x_{ij,a} - z_i} + (1-\epsilon)z_j - 2 - \frac{m^2}{p_i \cdot p_j}\right]$$

Motivation: $$\frac{1}{p_i \cdot p_j}\frac{p_a \cdot p_i}{p_a \cdot p_j + p_i \cdot p_j} - \frac{m_i^2}{2(p_i \cdot p_j)^2} = \frac{1}{p_i \cdot p_j}\left(\frac{2 - x_{ij,a}}{2 - x_{ij,a} - z_i} - 1\right) - \frac{m_i^2}{2(p_i \cdot p_j)^2}$$

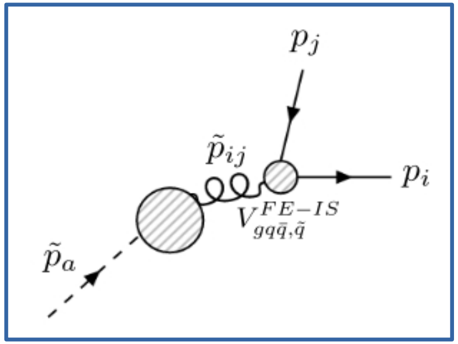


Final state emission of a gluon off a gluon, with the initial state squark as spectator

$$V^{FE-IS}_{gg,\tilde{q}} = 16\pi\alpha_s C_a N_c \mu^{2\epsilon}\left[\frac{1}{2 - x_{ij,a} - z_j} + \frac{1}{(1-x)+z_j} - 2 + (z_+ - z_j)(z_j - z_-)\right]$$

Motivation: $x_{ij,a} \xrightarrow{collinear} 1 \quad \Rightarrow V^{FE-IS}_{gg,\tilde{q}} \xrightarrow{collinear} 16\pi\alpha_s C_a N_c \mu^{2\epsilon}\left[\frac{1}{1-z_j} + \frac{1}{z_j} - 2 + (z_+ - z_j)(z_j - z_-)\right]$

Compare to: $P_{gg} = 2C_a\left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right]$



Final state gluon decay into a quark-antiquark pair, with the initial squark as spectator

$$V^{FE-IS}_{gq\bar{q},\tilde{q}} = 8\pi\alpha_s T_r N_c\left[1 - \frac{2}{1-\epsilon}(z_+ - z_j)(z_j - z_-)\right]$$

With:

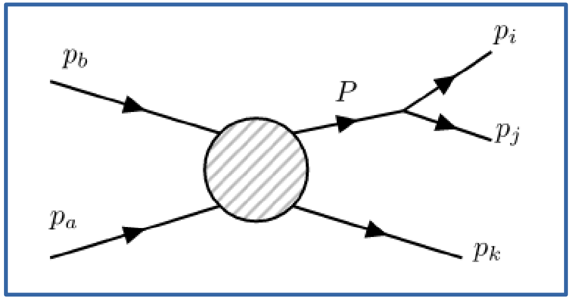
$$z_\pm(x) = \frac{1}{2}(1 \pm R_{ai}(x))$$

$$R_{ai}(1) = 1$$

Compare to: $P_{gq}(z,\epsilon) = T_r\left[1 - \frac{2z(1-z)}{1-\epsilon}\right]$

Phase Space Factorization

In order to integrate the dipoles, we need to separate the **three-particle phase space** into a **dipole phase space** and a **two-particle phase space (corresponding to the tree matrix element)**



Start by **factorizing** the **three-particle phase space** in the **normal momenta:**

$$d\Phi_3(p_a+p_b;p_i,p_j,p_k) = dP^2 d\Phi_2(p_a+p_b,P,p_k)d\Phi(P,p_i,p_j)$$

$$d\Phi(P,p_i,p_j) = \frac{1}{(4\pi)^{2-\epsilon}}\frac{(P^2)^{-\epsilon}}{\Gamma(1-\epsilon)}\left(\frac{\sqrt{\lambda_{ai}}R_{ai}(x)}{-\bar{Q}^2}\right)^{-1+2\epsilon} dz_i\left[(z_+-z_i)(z_i-z_-)\right]^{-\epsilon}$$

$$d\Phi_2(p_a+p_b;P,p_k) = \frac{1}{8\pi}dQ^2\frac{1}{\sqrt{\lambda(s,m_a^2,m_b^2)}}d\Omega_2$$

$$d\tilde{\Phi}_2(\tilde{p}_a+p_b;\tilde{p}_{ij},p_k) = \frac{1}{8\pi}dQ^2\frac{1}{\sqrt{\lambda(\tilde{s},m_a^2,m_b^2)}}d\Omega_2$$

with $Q^2 = (\tilde{p}_{ij}-\tilde{p}_a)^2 = (p_b-p_k)^2$

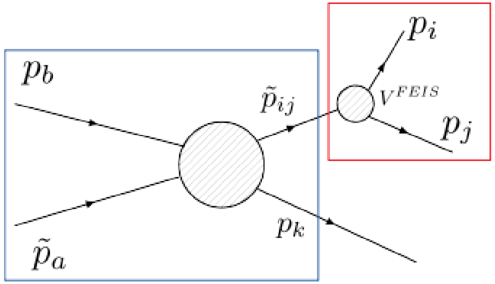
We now use the substitution: $x = x_{ij,a} = \dfrac{-Q^2 + m_a^2 + m_i^2}{P^2 - Q^2 + m_a^2}$

$$\Longrightarrow d\Phi_3(p_a + p_b; p_i, p_j, p_k) = dx \;\; d\tilde{\Phi}_2(\tilde{p}_a + p_b; \tilde{p}_{ij}, p_k) \frac{\sqrt{\lambda(\tilde{s}, m_a^2, m_b^2)}}{\sqrt{\lambda(s, m_a^2, m_b^2)}} \left[\left| \frac{\partial P^2}{\partial x} \right| d\Phi(P; p_i, p_j) \right]$$

For simplicity define: $$d\Phi_{dipole} = \left[\frac{\sqrt{\lambda(\tilde{s}, m_a^2, m_b^2)}}{\sqrt{\lambda(s, m_a^2, m_b^2)}} \left| \frac{\partial P^2}{\partial x} \right| d\Phi(P; p_i, p_j) \right]$$

$$\boxed{d\Phi_3(p_a + p_b; p_i, p_j, p_k) = dx \;\; d\tilde{\Phi}_2(\tilde{p}_a + p_b; \tilde{p}_{ij}, p_k) d\Phi_{dipole}}$$

Integrated over two-particle phase space.



Integrated over dipole phase space.

Integration

To cancel the virtual divergences we need to perform the integral:

$$\int d\Phi_3 \;\; D^{FE-IS} = \int dx \;\; d\tilde{\Phi}_2(Q^2, x)\left[d\Phi_{dipole} D^{FE-IS}\right]$$

for all dipole splitting functions.

Start with $\int d\Phi_{dipole} D^{FE-IS}$ Involves non trivial integration over variable z:

e.g the case of emission from quark: $$\int_{z_-}^{z_+} dz \left[\frac{2}{1-x-z} - 2 + (1-\epsilon)(1-z) - \frac{2m_i^2}{p_i \cdot p_j}\right]$$

Result can be written in the generic form:

$$\int d\Phi_{dipole} D^{FE-IS} = f(x;\epsilon) K(x;\epsilon)$$

Function **f(x;ε)** is divergent at x=1

Function **K(x;ε)** is finite for all x

$\Longrightarrow$

Perform the integration of the remaining part over x using the “+”-distribution trick

$$f(x;\epsilon)K(x;\epsilon) = \Big[f(x;0)\Big]^{+}_{[x_{min},1]} K(x;0) + \delta(1-x)\left[\int_{x_{min}}^{1} dx f(x;\epsilon)\right] K(x;\epsilon)$$

The **generic result** for all dipoles is then:

$$\int d\Phi_3 D_{ij,a}^{FE-IS} = \int dx \;\; d\tilde{\Phi}_2(x, Q^2) \left[f_{ij,a}(x;0) \right]^{+}_{[x_{min},1]} K_{ij,a}(x, Q^2; \epsilon)$$

$$+ \int d\tilde{\Phi}_2(1, Q^2) \left[\int_{x_{min}}^{1} dy \;\; f_{ij,a}(y;\epsilon) \right] K_{ij,a}(1, Q^2; \epsilon)$$

After expanding up to O(ε), we can write the result in the generic form:

$$\int d\Phi_3 D_{ij,a}^{FE-IS} = \boxed{\int dx \;\; d\tilde{\Phi}_2(x, Q^2) \;\; I_{finite,ija}^{FE-IS}(x, Q^2) \;\; |\tilde{M}_{tree}|^2}$$

Finite contribution

$$+ \boxed{\int d\Phi_2 \;\; I_{pole,ija}^{FE-IS}(\epsilon) \;\; |M_{tree}|^2}$$

Cancels IR virtual divergences

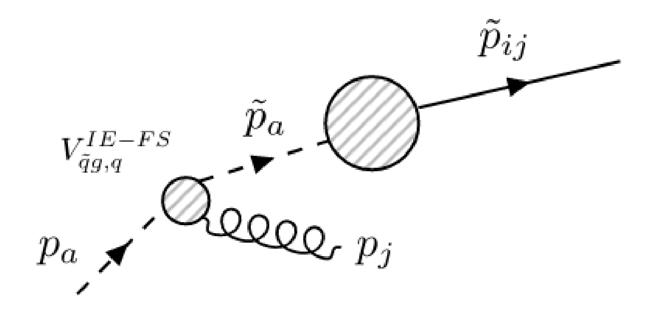
I_{pole} is a sum over terms in inverse powers of ε.
The second term will cancel IR divergences in the virtual contribution.

$$I_{pole,ija}^{FE-IS}(\epsilon) = \frac{a_1}{\epsilon^2} + \frac{b_1}{\epsilon} + c_1'$$

Initial state emission - Final state spectator

- Dipole kinematics **(same as the case of final emission – initial spectator)**
- Dipole Splitting function **(consider the dipoles related to squarks)**
- Phase space factorization **(same as the case of final emission – initial spectator)**
- Integration **(same concepts as the case of final emission – initial spectator)**

Dipole splitting functions



Initial state emission of a gluon off a squark, with final state quark/gluon as spectator

$$V^{IE-FS}_{\tilde{q},gq} = 8\pi\alpha_s C_f N_c \left[\frac{2}{2 - x_{\tilde{q},gq} - z_q} - 2 - \frac{m^2_{\tilde{q}}}{p_a \cdot p_j}\right]$$

Recall:

$$x_{\tilde{q},gq} = \frac{p_{\tilde{q}} \cdot p_q + p_{\tilde{q}} \cdot p_g - p_q \cdot p_g}{p_{\tilde{q}} \cdot p_q + p_{\tilde{q}} \cdot p_g} \qquad z_q = \frac{p_{\tilde{q}} \cdot p_q}{p_{\tilde{q}} \cdot p_q + p_{\tilde{q}} \cdot p_g}$$

Motivation: $\frac{1}{p_a \cdot p_j} \frac{p_a \cdot p_i}{p_a \cdot p_j + p_i \cdot p_j} - \frac{m_a^2}{2(p_a \cdot p_j)^2} = \frac{1}{p_a \cdot p_j}\left(\frac{2 - x_{ij,a}}{2 - x_{ij,a} - z_i} - 1\right) - \frac{m_a^2}{2(p_i \cdot p_j)^2}$

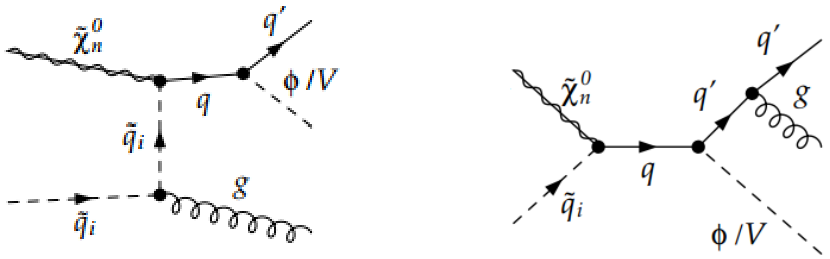
Implementation of the dipoles

Recall:

$$\sigma_{NLO} = \int d\Phi_3 \left[(d\sigma_{real})_{\epsilon=0} - (d\sigma_A)_{\epsilon=0} \right] + \int d\Phi_2 \;\; d\sigma_{virtual} + \int d\Phi_3 \;\; d\sigma_A$$

$$= \int d\Phi_3 \left[(d\sigma_{real})_{\epsilon=0} - (d\sigma_A)_{\epsilon=0} \right] + \int d\Phi_2 \;\; d\sigma_{virtual} + \int dx \;\; d\tilde{\Phi}_2(x) \int d\Phi_{dipole} \;\; d\sigma_A$$

We use the formalism to cancel all IR divergences in NLO of **Neutralino-Stop coannihilation into a top quark and a higgs boson.**



For these processes, we need to include **final state emission-initial state spectator** dipole $D^{FE\text{-}IS}$ and **Initial state emission-Final state spectator** dipole $D^{IE\text{-}FS}$.

$$d\sigma_A = D^{FE-IS} + D^{IE-FS}$$

The dipoles are given by:

$$D_{\tilde{q}g,q}^{IE-FS} = -\frac{1}{2p_{\tilde{q}} \cdot p_g} \langle \tilde{p}_{\tilde{q}}, \tilde{p}_q | \frac{T_q \cdot T_{\tilde{q}}}{T_{\tilde{q}}^2} V_{\tilde{q}g,q}^{IE-FS} | \tilde{p}_{\tilde{q}}, \tilde{p}_q \rangle$$

$$D_{qg,\tilde{q}}^{FE-IS} = -\frac{1}{2p_q \cdot p_g} \langle \tilde{p}_{\tilde{q}}, \tilde{p}_q | \frac{T_{\tilde{q}} \cdot T_q}{T_q^2} V_{qg,\tilde{q}}^{FE-IS} | \tilde{p}_{\tilde{q}}, \tilde{p}_q \rangle$$

Express $|\tilde{M}_2|^2$ in terms of the dipole Mandelstam variables:

$$\tilde{s} = (p_\chi + \tilde{p}_{\tilde{q}})^2 \qquad \tilde{t} = (\tilde{q} - p_\chi)^2$$

$$\tilde{u} = (\tilde{p}_q - \tilde{p}_{\tilde{q}})^2 = Q^2$$

Now use the condition of color conservation: $\left(\sum_{initial} T_a + \sum_{final} T_f \right) |p_a, p_f\rangle = 0$

Applied to our case: $\left(T_{\tilde{q}} + T_q \right) |\tilde{p}_{\tilde{q}}, \tilde{p}_q\rangle = 0 \Longrightarrow T_{\tilde{q}} \, |\tilde{p}_{\tilde{q}}, \tilde{p}_q\rangle = -T_q \, |\tilde{p}_{\tilde{q}}, \tilde{p}_q\rangle$

$$\Longrightarrow \frac{T_q \cdot T_{\tilde{q}}}{T_{\tilde{q}}^2} \, |\tilde{p}_{\tilde{q}}, \tilde{p}_q\rangle = -\frac{T_{\tilde{q}}^2}{T_{\tilde{q}}^2} \, |\tilde{p}_{\tilde{q}}, \tilde{p}_q\rangle$$

The dipole functions then become:

$$D_{\tilde{q}g,q}^{IE-FS} = \frac{1}{2p_{\tilde{q}} \cdot p_g} V_{\tilde{q}g,q}^{IE-FS} \left| \tilde{M}_2 \right|^2$$

$$D_{qg,\tilde{q}}^{FE-IS} = \frac{1}{2p_q \cdot p_g} V_{qg,\tilde{q}}^{FE-IS} \left| \tilde{M}_2 \right|^2$$

In this form, **subtracting the dipoles** from the real cross section cancels all divergences in the **real contribution.**

Consider now the implementation in the virtual contribution:

Recall that the general form of the integrated dipoles is:

$$\int d\Phi_3 (D^{FE-IS}_{qg,\tilde{q}} + D^{IE-FS}_{\tilde{q}g,q}) = \int dx \;\; d\tilde{\Phi}_2(x,Q^2) \;\; \left(I^{FE-IS}_{finite,qg\tilde{q}}(x,Q^2) + I^{IE-IS}_{finite,\tilde{q}gq}(x,Q^2)\right) \;\; |\tilde{M}_{tree}|^2$$

$$\boxed{+ \int d\Phi_2 \;\; (I^{FE-IS}_{pole,qg\tilde{q}}(x,Q^2) + I^{IE-FS}_{pole,\tilde{q}gq}(x,Q^2)) \;\; |M_{tree}|^2}$$

Will cancel all the divergences in the virtual contribution

The other x-dependent part will be integrated over the two-particle phase space in the dipole momenta.

$$\int dx \int d\tilde{\Phi}_2(x) I_{finite}(x) |\tilde{M}_2|^2(x) = \int_{x_{min}}^{1} dx \int_{Q^2_{min}(x)}^{Q^2_{max}} dQ^2 \;\; I_{finite}(x,Q^2) \left|\tilde{M}_2\right|^2$$

Results

Start with the real contribution

$$d\sigma_{real} = |M_3|^2 \xrightarrow{soft \quad limit} (\frac{1}{S_a}V_a|M_2|^2 + \frac{1}{S_3}V_3|M_2|^2)$$

Dipole counterpart

$$D^{FE-IS} = \frac{1}{S_3}V_3^{FE-IS}|\tilde{M}_2|^2 \xrightarrow{soft \quad limit} \frac{1}{S_3}V_3|M_2|^2$$

$$D^{IE-FS} = \frac{1}{S_a}V_a^{FE-IS}|\tilde{M}_2|^2 \xrightarrow{soft \quad limit} \frac{1}{S_a}V_a|M_2|^2$$

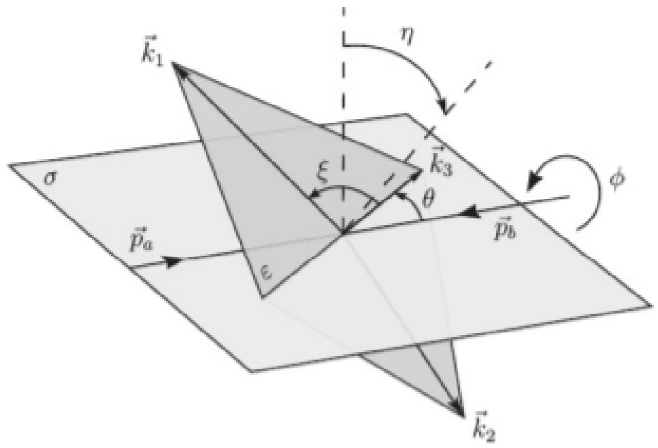
$$\int d\Phi_3 = \frac{s}{32}\frac{1}{(2\pi)^5}\int_0^{x_{3max}} dx_3 \int_{x_{1min}(x_3)}^{x_{1max}(x_3)} dx_1 \int d\eta \int d\cos(\theta) \int d\phi$$

Where:

$$x_3 = \frac{E_{gluon}}{\sqrt{s}}$$

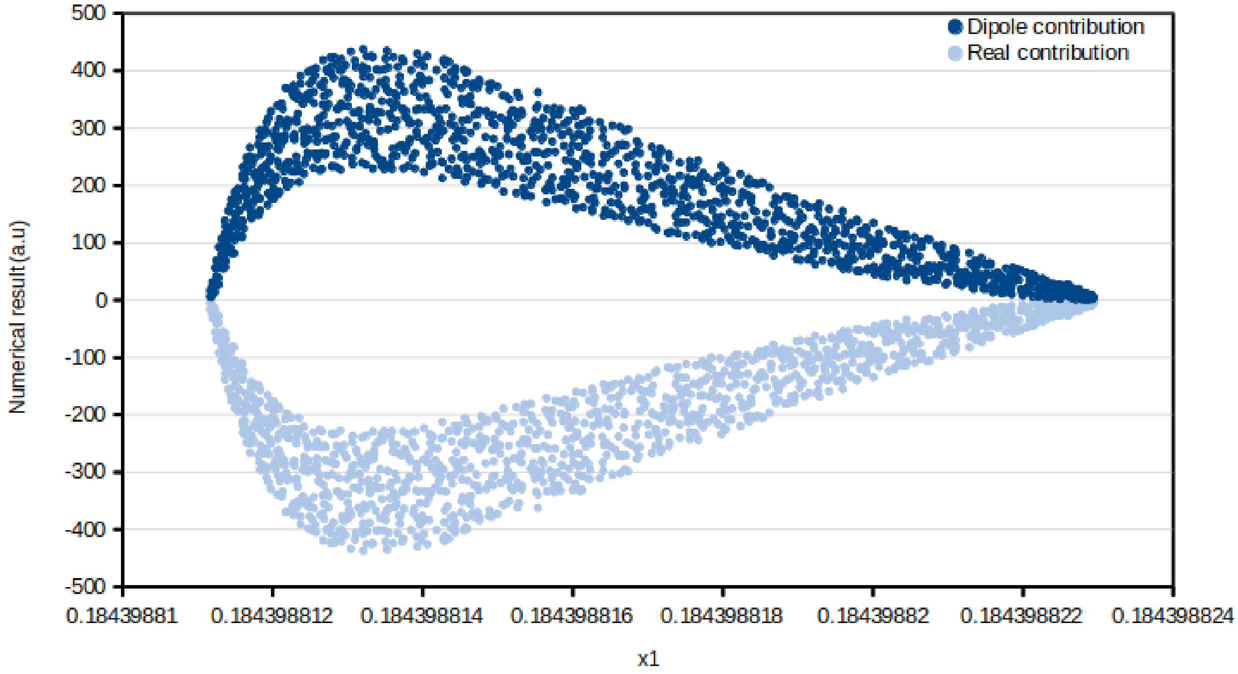
$$x_1 = \frac{E_{top}}{\sqrt{s}}$$

$$x_2 = \frac{E_{higgs}}{\sqrt{s}}$$



Source: Hitchhiker's guide to renormalization

Fix x_3 to a very small value to get soft limit: $x_3 = 0.000001$

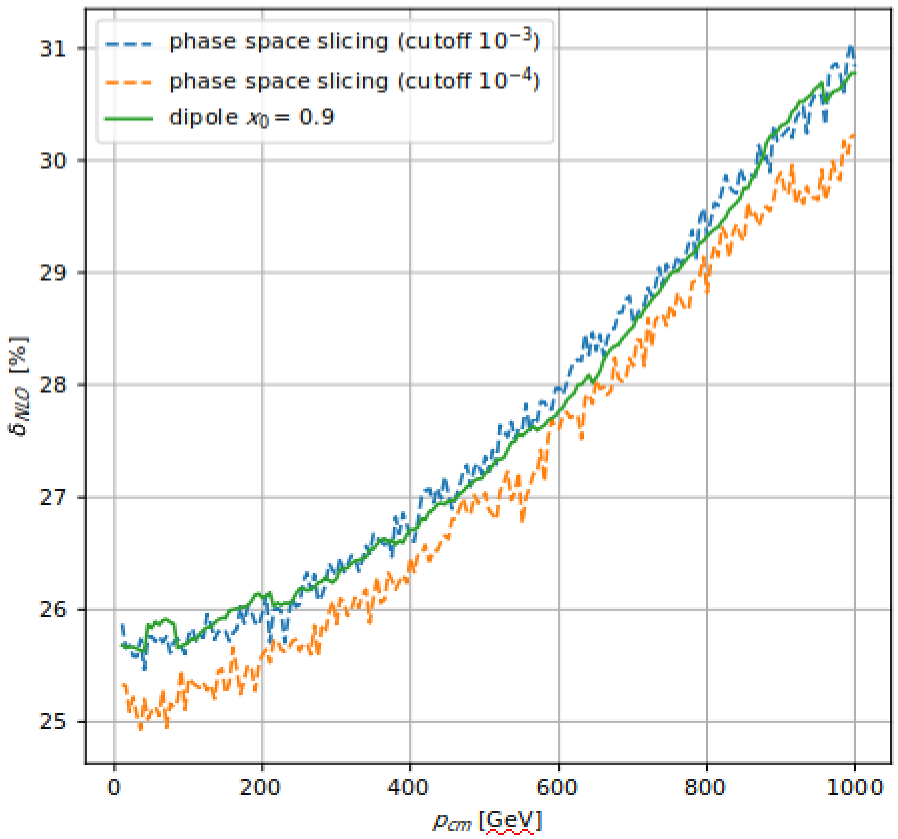


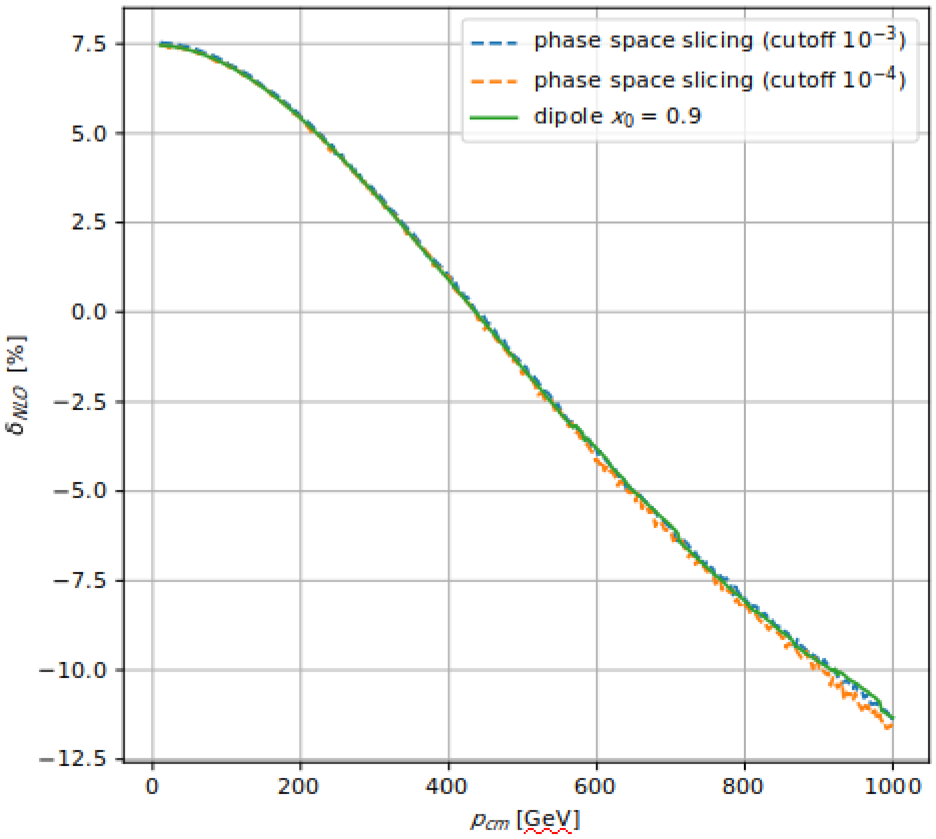
Here dipole contribution multiplied with a minus sign.

We see that there is good agreement between both contributions and all the IR divergences cancel.

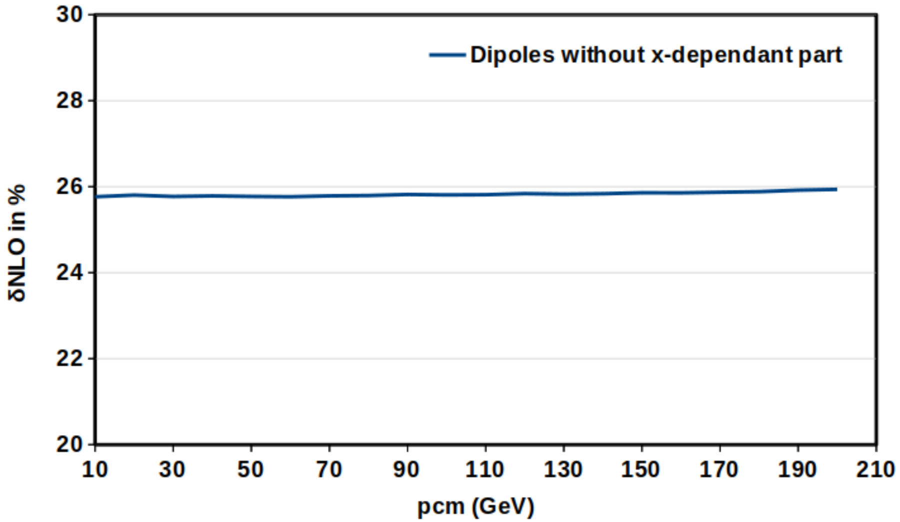
Full Results

Work with Luca Wiggering (Münster) after submitting the thesis

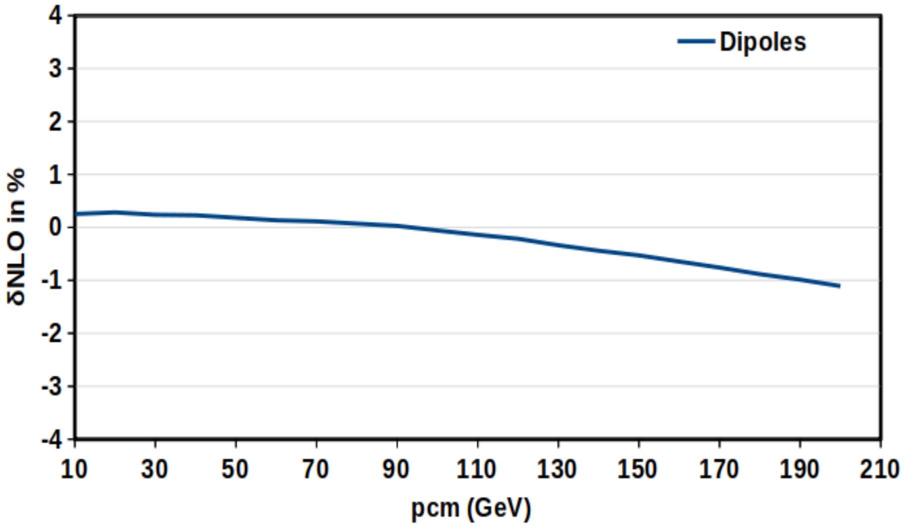




Full result after removing the divergences in the virtual part and real part.

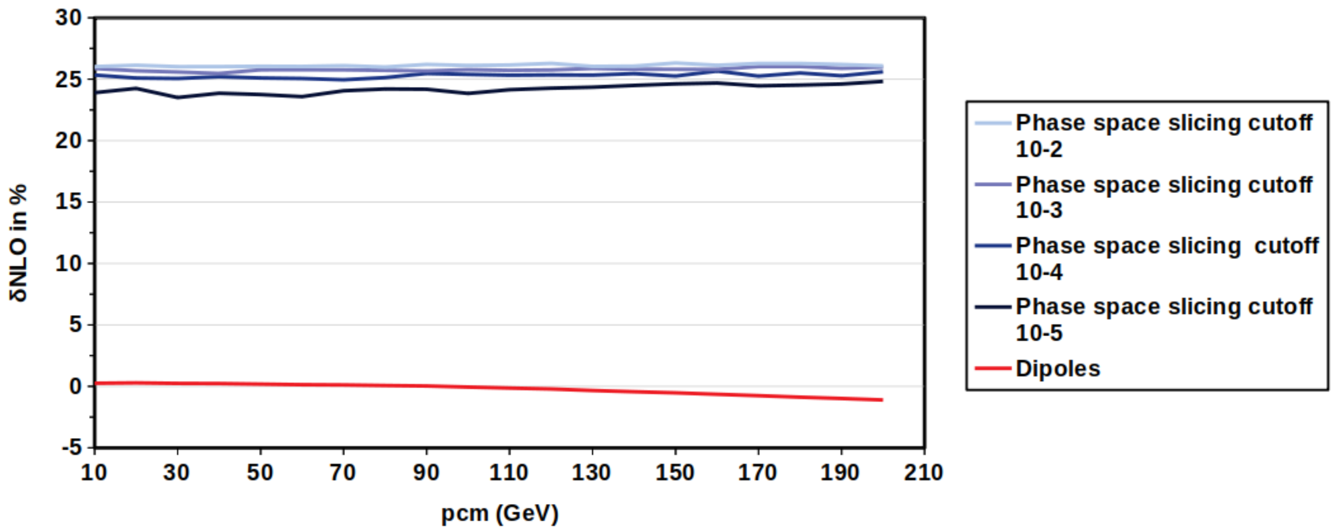


Full implementation of the dipoles.



The contribution from the x-dependent part is very high.

Compare the results with phase space slicing method:



High discrepency with the results from phase space slicing!

The issue seems to be related to the integration result of the x-dependent part as it gives big contributions.

$$\int dx \int d\tilde{\Phi}_2(x) I_{finite}(x)|\tilde{M}_2|^2(x) = \int_{x_{min}}^{1} dx \int_{Q^2_{min}(x)}^{Q^2_{max}} dQ^2 \;\; I_{finite}(x, Q^2) \left|\tilde{M}_2\right|^2$$

Conclusions and outlook

- We developed the QCD dipole formalism for **massive initial state particles**, including the possibility of **scalars as emitters.** All this using **dimensional regularization**.
- The formalism was implemented for the NLO QCD corrections to the coannihilation process of neutralino-stop into a top quark and a higgs boson. The formalism successfully canceled **all IR divergences in the real as well in the virtual contribution**.
- The results were compared to the results using **phase space slicing**. We observe a very good agreement with the optimized results of phase space slicing.
- Implementation in the case of a **final state gluon** is in progress. All divergences are cancelled but there is a **discrepency between the results from the dipoles and phase space slicing**.

Backup

Virtual Contribution

$$\sigma_{virtual} = \int d\Phi_2 \ \ |M_2|^2_{loop}$$
$$= \int d\Phi_2 \ \ (\frac{a}{\epsilon^2} + \frac{b}{\epsilon} + c)|M_2|^2$$

Dipole counterpart:

$$\int d\Phi_3 \ \ d\sigma_A = \int dx \ \ \int d\tilde{\Phi}_2(x) \int d\Phi_{dipole} \ \ d\sigma_A$$

$$\int d\Phi_{dipole} \ \ d\sigma_A = I_{finite}(x)|\tilde{M}_2|^2(x) + \delta(1-x)I_{pole}|\tilde{M}_2|^2(x)$$

$$\Longrightarrow \int d\Phi_3 \ \ d\sigma_A = \int dx \ \ \int d\tilde{\Phi}_2(x) \ \ I_{finite}(x)|\tilde{M}_2|^2(x)$$
$$+ \int d\tilde{\Phi}_2(1) \ \ I_{pole}|M_2|^2(1)$$

Phase space slicing (Backup)

$$d\tilde{\Phi}_2(\tilde{p}_a + p_b; \tilde{P}_{ab}) \quad dP_{ab}^2 \quad d\tilde{\Phi}_2(\tilde{P}_{ab}; \tilde{p}_{k1}, \tilde{p}_{k2})$$

$$= dP_{ab}^2 (2\pi)^d \delta^{(d)}(\tilde{P}_{ab} - \tilde{p}_a - p_b) \frac{d^d \tilde{P}_{ab}}{(2\pi)^{d-1}} \delta_+(\tilde{P}_{ab}^2 - P_{ab}^2)(2\pi)^d \delta^{(d)}(\tilde{P}_{ab} - \tilde{p}_{k1} - \tilde{p}_{k2}) \frac{d^d \tilde{p}_{k1}}{(2\pi)^{d-1}} \frac{d^d \tilde{p}_{k2}}{(2\pi)^{d-1}}$$

$$= (2\pi) dP_{ab}^2 \delta_+(\tilde{P}_{ab}^2 - P_{ab}^2)(2\pi)^d \delta^{(d)}(\tilde{p}_a + p_b - \tilde{p}_{k1} - \tilde{p}_{k2}) \frac{d^d \tilde{p}_{k1}}{(2\pi)^{d-1}} \frac{d^d \tilde{p}_{k2}}{(2\pi)^{d-1}}$$

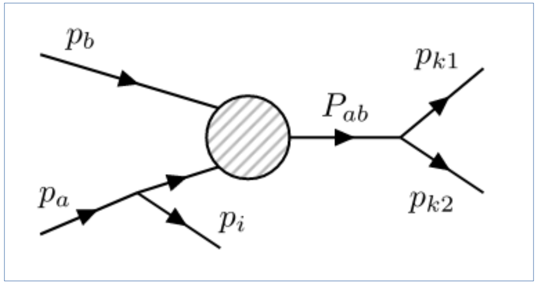
$$= (2\pi) dP_{ab}^2 \delta_+(\tilde{P}_{ab}^2 - P_{ab}^2) d\tilde{\Phi}_2(\tilde{p}_a + p_b; \tilde{p}_{k1}, \tilde{p}_{k2})$$

Initial state emitter – initial state spectator

- Dipole Kinematics
- Dipole splitting functions
- Phase space factorization
- Integration

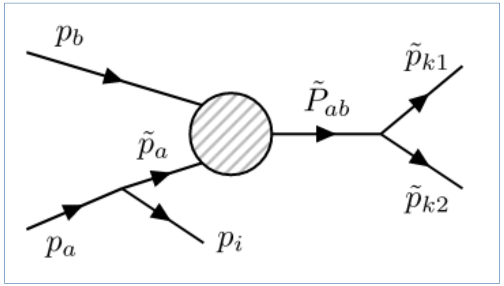
Dipole Kinematics

We need to define new momenta in order to have well defined matrix elements in all points of the phase space.



Usual momentum conservation:

$$p_a + p_b = p_{k1} + p_{k2} + p_i$$



Dipole momenta conservation:

$$\tilde{p}_a + p_b = \tilde{P}_{ab} = \tilde{p}_{k1} + \tilde{p}_{k2}$$

Postulate the following dipole momenta:

$$\tilde{p}_a^\mu = f(P_{ab}^2, m_a^2, m_b^2) p_a^\mu + g(P_{ab}^2, m_a^2, m_b^2) p_b^\mu$$

$$\tilde{P}_{ab}^\mu = \tilde{p}_a^\mu + p_b^\mu = f(P_{ab}^2, m_a^2, m_b^2) p_a^\mu + \big(g(P_{ab}^2, m_a^2, m_b^2) + 1\big) p_b^\mu$$

The dipole momenta should satisfy the conditions:

Momentum conservation:

$$\tilde{p}_a + p_b = \tilde{P}_{ab} = \tilde{p}_{k1} + \tilde{p}_{k2}$$

On-shell relations:

$$\tilde{p}_a^2 = m_a^2, \tilde{P}_{ab}^2 = P_{ab}^2$$

Soft limit:

$$\tilde{p}_a \xrightarrow{soft} p_a, \tilde{P}_{ab}^\mu \xrightarrow{soft} P_{ab}^\mu$$

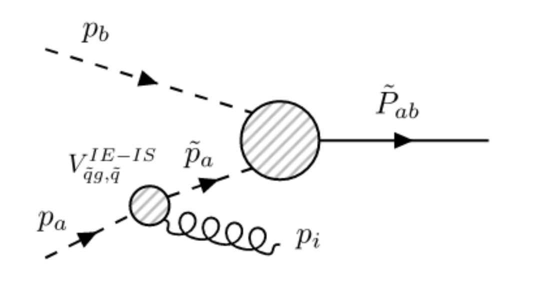
We get a system of two equations for f and g:

$$m_a^2 = f^2 m_a^2 + 2fg p_a \cdot p_b + g^2 m_b^2$$
$$P_{ab}^2 = f^2 m_a^2 + 2f(g+1) p_a \cdot p_b + (g+1)^2 m_b^2$$

Solving for f and g:

$$\tilde{p}_a^\mu = \frac{\sqrt{\lambda(P_{ab}^2, m_a^2, m_b^2)}}{\sqrt{\lambda(s, m_a^2, m_b^2)}} p_a^\mu + \left(\frac{P_{ab}^2 - m_a^2 + m_b^2}{2m_b^2} - \frac{p_a \cdot p_b}{m_b^2} \frac{\sqrt{\lambda(P_{ab}^2, m_a^2, m_b^2)}}{\sqrt{\lambda(s, m_a^2, m_b^2)}} \right) p_b^\mu$$
$$\tilde{p}_{ab}^\mu = \frac{\sqrt{\lambda(P_{ab}^2, m_a^2, m_b^2)}}{\sqrt{\lambda(s, m_a^2, m_b^2)}} p_a^\mu + \left(\frac{P_{ab}^2 - m_a^2 + m_b^2}{2m_b^2} - \frac{p_a \cdot p_b}{m_b^2} \frac{\sqrt{\lambda(P_{ab}^2, m_a^2, m_b^2)}}{\sqrt{\lambda(s, m_a^2, m_b^2)}} + 1 \right) p_b^\mu$$

Dipole splitting functions



Initial state emission of a gluon off a squark, with a squark in the initial state as spectator.

$$V^{IE-IS}_{\tilde{q}g,\tilde{q}} = \frac{8\pi\alpha_s C_f N_c \mu^{2\epsilon}}{x_{ab}} \left[\frac{2}{1-x_{ab}} - 2 - \frac{x_{ab} m_a^2}{p_a \cdot p_i} \right]$$

where $$x_{ab} = \frac{p_a \cdot p_b - p_a \cdot p_i - p_b \cdot p_i}{p_a \cdot p_b}$$

$$|M_3|^2 \xrightarrow{soft} -4\pi\alpha_s C_f N_c \mu^{2\epsilon} \left[\frac{m_a^2}{p_a \cdot p_i} + \frac{m_b^2}{p_b \cdot p_i} - \frac{2 p_a \cdot p_b}{(p_a \cdot p_i)(p_b \cdot p_i)} \right]$$

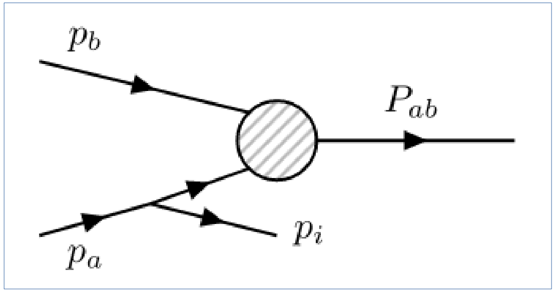
will be canceled with:

$$D^{IE-IS}_{\tilde{q}_1 g,\tilde{q}_2} + D^{IE-IS}_{\tilde{q}_2 g,\tilde{q}_1} = \left(\frac{1}{2p_a \cdot p_i} V^{IE-IS}_{\tilde{q}_1 g,\tilde{q}_2} + \frac{1}{2p_b \cdot p_i} V^{IE-IS}_{\tilde{q}_2 g,\tilde{q}_1}\right) \left|\tilde{M}_2\right|^2$$

$$
\begin{aligned}
D^{IE-IS}_{\tilde{q_1}g,\tilde{q_2}} + D^{IE-IS}_{\tilde{q_2}g,\tilde{q_1}} &= \left(\frac{1}{2p_a\cdot p_i}V^{IE-IS}_{\tilde{q_1}g,\tilde{q_2}} + \frac{1}{2p_b\cdot p_i}V^{IE-IS}_{\tilde{q_2}g,\tilde{q_1}}\right)\left|\tilde{M}_2\right|^2 \\
&= \frac{8\pi\alpha_s C_f N_c \mu^{2\epsilon}}{x_{ab}}\left[-\frac{x_{ab}m_a^2}{2(p_a\cdot p_i)^2} - \frac{x_{ab}m_b^2}{2(p_b\cdot p_i)^2} + \left(\frac{1}{2p_a\cdot p_i} + \frac{1}{2p_b\cdot p_i}\right)\left(\frac{2}{1-x_{ab}} - 2\right)\right]\left|\tilde{M}_2\right|^2 \\
&= \frac{8\pi\alpha_s C_f N_c \mu^{2\epsilon}}{x_{ab}}\left[-\frac{x_{ab}m_a^2}{2(p_a\cdot p_i)^2} - \frac{x_{ab}m_b^2}{2(p_b\cdot p_i)^2} + \frac{x_{ab}p_a\cdot p_b}{(p_a\cdot p_i)(p_b\cdot p_i)}\right]\left|\tilde{M}_2\right|^2
\end{aligned}
$$

$$
\xrightarrow{soft} 4\pi\alpha_s C_f N_c \mu^{2\epsilon}\left[-\frac{m_a^2}{(p_a\cdot p_i)^2} - \frac{m_a^2}{(p_a\cdot p_i)^2} + \frac{2p_a\cdot p_b}{(p_a\cdot p_i)(p_b\cdot p_i)}\right]\left|M_2\right|^2
$$

Phase space factorization



We separate the three-particle phase space into a two particle phase space and a dipole phase space:

$$d\Phi_3(p_a + p_b; p_i, P_{ab}) = dx \quad d\tilde{\Phi}_2\left(\tilde{p}_a(x) + p_b; \tilde{P}_{ab}(x)\right) \quad d\Phi_{dipole}$$

Calculate $d\Phi_3(p_a + p_b; p_i, P_{ab})$ explicitly:

$$d\Phi_3(p_a + p_b; p_i, P_{ab}) = \frac{s_{ab}^{1-2\epsilon}}{4(\pi)^{2-2\epsilon}} \frac{s^{-\epsilon}}{\sqrt{\lambda_{ab}}^{1-2\epsilon}} d\Omega_{2-2\epsilon} dy_{ab} [(y_+ - y_{ab})(y_{ab} - y_-)]^{-\epsilon}$$

$$\begin{aligned} d\tilde{\Phi}_2(\tilde{p}_a + p_b; \tilde{P}_{ab}) &= (2\pi)^d \delta^d(\tilde{P}_{ab}^{\mu} - \tilde{p}_a - p_b) \frac{d^d \tilde{P}_{ab}}{(2\pi)^{d-1}} \delta_+(\tilde{P}_{ab}^2 - P_{ab}^2) \\ &= (2\pi)\delta_+(\tilde{P}_{ab}^2 - P_{ab}^2) \\ &= (2\pi) \left| \frac{\partial \tilde{P}_{ab}^2}{\partial x} \right|^{-1} \delta_+(x - x_{ab}) \end{aligned}$$

$$d\Phi_3(p_a+p_b;P_{ab},p_i) = dx \quad \delta_+(x-x_{ab}) \quad d\Phi_3(p_a+p_b;P_{ab},p_i)$$

$$= dx \quad \frac{1}{2\pi}\left|\frac{\partial \tilde{P}_{ab}^2}{\partial x}\right| d\tilde{\Phi}_2(\tilde{p}_a+p_b;\tilde{P}_{ab}) \quad d\Phi_3(p_a+p_b;P_{ab},p_i)$$

now use:

$$d\Phi_3(p_a+p_b;p_i,P_{ab}) = \frac{s_{ab}^{1-2\epsilon}}{4(\pi)^{2-2\epsilon}}\frac{s^{-\epsilon}}{\sqrt{\lambda_{ab}}^{1-2\epsilon}} d\Omega_{2-2\epsilon} dy_{ab}[(y_+-y_{ab})(y_{ab}-y_-)]^{-\epsilon}$$

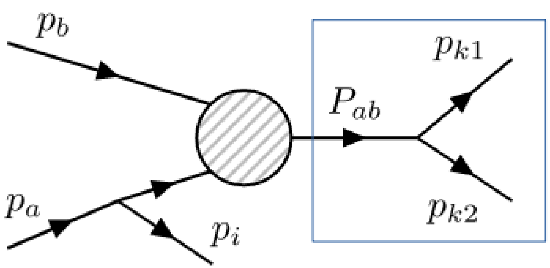
and define:

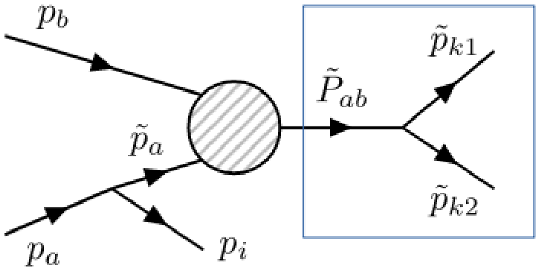
$$d\Phi_{dipole} = \frac{1}{2\pi}\left|\frac{\partial \tilde{P}_{ab}^2}{\partial x}\right|\frac{s_{ab}^{1-2\epsilon}}{4(\pi)^{2-2\epsilon}}\frac{s^{-\epsilon}}{\sqrt{\lambda_{ab}}^{1-2\epsilon}} d\Omega_{2-2\epsilon} dy_{ab}[(y_+-y_{ab})(y_{ab}-y_-)]^{-\epsilon}$$

We end up with:

$$d\Phi_3(p_a+p_b;p_i,P_{ab}) = dx \quad d\tilde{\Phi}_2\left(\tilde{p}_a(x)+p_b;\tilde{P}_{ab}(x)\right) \quad d\Phi_{dipole}$$

For actual implementations, we need the phase space with particles $\tilde{p}_{k1}$ and $\tilde{p}_{k2}$





$$d\Phi_3(p_a+p_b;p_i;P_{ab}) \;\; dP_{ab}^2 \;\; \left[(2\pi)^d \frac{d^d p_{k1}}{(2\pi)^{d-1}} \frac{d^d p_{k2}}{(2\pi)^{d-1}} \delta^d(P_{ab}-p_{k1}-p_{k2})\right]$$

$$= (2\pi) d\Phi_3(p_a+p_b;p_i,p_{k1},p_{k2})$$

$$\Longrightarrow d\Phi_3(p_a+p_b;p_i,p_{k1},p_{k2}) = \frac{1}{2\pi} dx \;\; d\tilde{\Phi}_2(\tilde{p}_a+p_b;\tilde{P}_{ab})\left[d\Phi_{dipole}\right] dP_{ab}^2 \;\; d\tilde{\Phi}_2(P_{ab};\tilde{p}_{k1},\tilde{p}_{k2})$$

use:

$$d\tilde{\Phi}_2(\tilde{p}_a+p_b;\tilde{P}_{ab}) \;\; dP_{ab}^2 \;\; d\tilde{\Phi}_2(\tilde{P}_{ab};\tilde{p}_{k1},\tilde{p}_{k2})$$
$$= (2\pi) \;\; dP_{ab}^2 \;\; \delta_+(\tilde{P}_{ab}^2 - P_{ab}^2) d\tilde{\Phi}_2(\tilde{p}_a+p_b;\tilde{p}_{k1},\tilde{p}_{k2})$$

$$\boxed{\Longrightarrow d\Phi_3(p_a+p_b;p_i,p_{k1},p_{k2}) = dx \;\; d\tilde{\Phi}_2(\tilde{p}_a+p_b;\tilde{p}_{k1},\tilde{p}_{k2})\left[d\Phi_{dipole}\right]}$$

Integration

The integration of the dipole splitting functions is analogous to the previous cases.

The general result of the integration over the dipole phase space is again given as a distribution in x.

$$\int d\Phi_{dipole} \ \ D^{IE-IS} = f^{IE-IS}(x;\epsilon) K^{IE-IS}(x;\epsilon)$$

divergent at x=1

finite for all x

Use “+”-distribution to write the result as a distribution in x:

$$\int d\Phi_3 D^{IE-IS} = \int dx \ \ d\tilde{\Phi}_2(x,\tilde{s}) \left[f^{IE-IS}(x;0) \right]^{+}_{[x_{min},1]} K^{IE-IS}(x;\epsilon)$$
$$+ \int d\tilde{\Phi}_2(1,s) \left[\int_{x_{min}}^{1} dy \ \ f^{IE-IS}(y;\epsilon) \right] K^{IE-IS}(1;\epsilon)$$

After expanding up to O(ε), we can write the result in the generic form:

$$\int d\Phi_3 D^{IE-IS} = \boxed{\int dx \;\; d\tilde{\Phi}_2(x,\tilde{s}) \;\; I^{IE-IS}_{finite}(x) \;\; |\tilde{M}_{tree}|^2}$$

Finite contribution

$$+ \boxed{\int d\Phi_2 \;\; I^{IE-IS}_{pole}(\epsilon) \;\; |M_{tree}|^2}$$

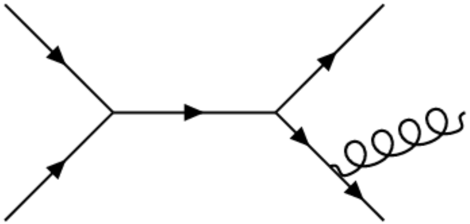
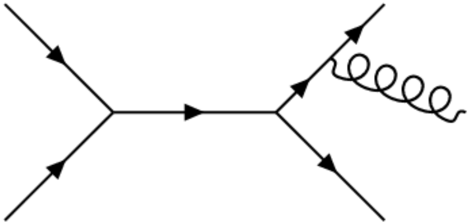
Cancels IR virtual divergences

I_{pole} is a sum over terms in inverse powers of ε.
The second term will cancel IR divergences in the virtual contribution.

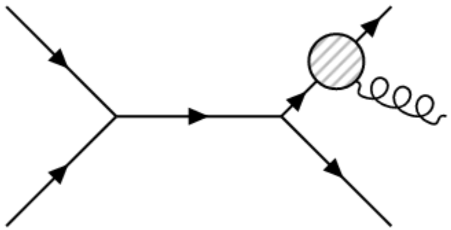
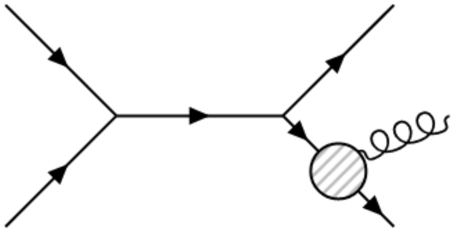
$$I^{IE-IS}_{pole}(\epsilon) = \frac{a_3}{\epsilon^2} + \frac{b_3}{\epsilon} + c_3'$$

Example

Real contribution

 $+$ 

$$\xrightarrow{soft\quad limit} \left(\frac{1}{S_1}V_{1,2} + \frac{1}{S_2}V_{2,1}\right)|M_2|^2$$

 $+$ 

$$= D_{13,2}^{FE-FS} + D_{23,1}^{FE-FS}$$

$$D_{13,2}^{FE-FS} = \frac{1}{S_1}\langle\tilde{p}_{13},\tilde{p}_2|\,\frac{T_{13}T_2}{T_2^2}V_{13,2}^{FE-FS}\,|\tilde{p}_{13},\tilde{p}_2\rangle = \frac{1}{S_1}V_{13,2}^{FE-FS}\left|\tilde{M}_2\right|^2$$