Universität Hamburg
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Klein–Gordon Quantum Field Theory in Curved Spacetime

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Agenda

- Hamiltonian Description of the KG-Field
  - Klein-Gordon field in curved spacetime
  - From Lagrangian to Hamiltonian
- Quantization and Discretizations
  - Lattice formalism and commutation relations
  - Momentum space
- Quantum Simulation
  - Mapping to Gaussian operations
  - Time evolution
  - Initial state
- Results
  - Friedmann-Lemaître-Robertson-Walker-Metric
  - Schwarzschild-Metric

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Klein-Gordon field in curved spacetime

Action

$$S[\phi] = \int d^4x \mathcal{L} = \int d^4x \sqrt{-g}\, \frac{1}{2} \left[g^{\mu\nu}\, \nabla_\mu \phi(x,t)\, \nabla_\nu \phi(x,t) - (m^2 + \xi R)\, \phi(x,t)^2 \right]$$

- Scalar field $\phi(x,t)$
- Background metric $g_{\mu\nu}(x,t)$
- Ricci scalar R with coupling strength ξ
- Covariant derivative $\nabla_\mu \phi = \partial_\mu \phi$

Varying the action with respect to the field yields the Klein–Gordon equation in curved spacetime

$$g^{\mu\nu} \partial_\nu \partial_\mu \phi(x,t) + (m^2 + \xi R) \phi(x,t) = 0$$

From Lagrangian to Hamiltonian

Hamiltonian

$$H = \int dx\, \mathcal{H} = \int dx \left(\dot{\phi}(x,t)\, \pi(x,t) \; - \; \mathcal{L} \right)$$

- Scalar field $\phi(x,t)$
- Conjugate Momentum $\pi(x,t) = \frac{\partial \mathcal{L}}{\partial_t \phi}$

in the following we will restrict to

- $g_{\mu\nu} = 0$ for $\mu \neq \nu$
- 1+1-dimensional spacetime

For this system the relation of field and its conjugate Momentum is

$$\partial_t \phi(x,t) \; = \; \frac{\pi(x,t)}{\sqrt{-g}\, g^{00}}$$

Hamiltonian

$$H = \frac{1}{2}\int dx \left(\frac{\pi(x,t)^2}{\sqrt{-g}\, g^{00}} - \sqrt{-g}\Big[g^{11}\Big(\partial_x \phi(x,t)\Big)^2 - m^2 \phi(x,t)^2 \Big] \right)$$

- Three constituents:
  - kinetic $\propto \pi(x,t)^2$
  - spacial gradient $\propto \Big(\partial_x \phi(x,t)\Big)^2$
  - mass $\propto \phi(x,t)^2$
- Quadratic in ϕ and $\pi \Rightarrow$ system of coupled harmonic oscillators
- Metric $g_{\mu\nu}(x,t)$ controls couplings and effective frequencies

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Quantization Promote variables to operators

- Field $\phi(x,t) \to \hat{\phi}(x,t)$
- conj. momentum $\pi(x,t) \to \hat{\pi}(x,t)$

Impose commutation relations:

$$[\phi(x), \pi(y)] = i\,\delta(x-y)$$

Discretization Move the System to a finite spatial Lattice where

- position $x \to x_n = an$
- Field $\phi(x,t) \to \phi_n$
- conj. momentum $\pi_n(x,t) \to \pi_n$
- Derivatives $\partial_x \phi \;\to\; \frac{\phi_{n+1}-\phi_n}{a}$
- Integration $\int dx \to a\sum_n^{N-1}$

For the following we introduce rescaled variables such that

$$\boxed{\frac{1}{\sqrt{a}}\,\hat{q}_n = \hat{\phi}_n, \qquad \frac{1}{\sqrt{a}}\,\hat{p}_n = \hat{\pi}_n}$$

Lattice formalism and commutation relations

Lattice Hamiltonian

$$H = \frac{1}{2}\sum_{n=0}^{N-1}\left[\frac{\hat{p}_n^2}{\sqrt{-g_n}\, g_n^{00}} + \sqrt{-g_n}\left(m^2 - \frac{2}{a^2}\, g_n^{11}\right)\hat{q}_n^2 + \frac{2}{a^2}\, g_n^{11}\sqrt{-g_n}\,\hat{q}_{n+1}\hat{q}_n\right]$$

- Three constituents:
  - kinetic $\propto \hat{p}_n^2$
  - "hopping" $\propto \hat{q}_{n+1}\hat{q}_n$
  - mass $\propto \hat{q}_n^2$
- Metric $g_{\mu\nu}(x,t)$ controls couplings and effective frequencies

The system reduces to a finite chain of coupled harmonic oscillators

Momentum space

Lattice Hamiltonian in momentum space for a translation-invariant metric ($g_n(t) = g(t)$)

$$H = \frac{1}{2}\sum_{k}^{N-1}\left[\frac{1}{\sqrt{-g}g^{00}}\hat{p}_k\hat{p}_{-k} + \sqrt{-g}\left(m^2 - \frac{2}{a^2}g^{11}\left\{1 - \Re\left(e^{2\pi i k/N}\right)\right\}\right)\hat{q}_k\hat{q}_{-k}\right]$$

- Diagonal in k $\Rightarrow$ independent modes
- Each mode is a harmonic oscillator with frequency ω_k :

$$\boxed{\omega_k^2 = \frac{1}{g^{00}}\left(m^2 - \frac{2}{a^2}g^{11}\left\{1 - \Re\left(e^{2\pi i k/N}\right)\right\}\right)}$$

- for $g_{\mu\nu} \to \eta_{\mu\nu}$ and taking the continuum limit

$$\omega_k^2 = m^2 + \frac{4}{a^2}\sin^2\left(\frac{\pi k}{N}\right) \xrightarrow{N\to\infty} \omega_k^2 = m^2 + p_k^2$$

Quantum Simulation

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Mapping to Gaussian operations

Lattice Hamiltonian

$$H = \frac{1}{2}\sum_{n=0}^{N-1}\left(\frac{\Lambda_n}{\sqrt{-g_n}\,g_n^{00}}\left(\hat{\mathbf{p}}_n^2 + \hat{\mathbf{q}}_n^2\right) + \frac{2}{a^2}\frac{g_n^{11}\sqrt{-g_n}}{\sqrt{\Lambda_n}\sqrt{\Lambda_{n+1}}}\,\hat{\mathbf{q}}_{n+1}\hat{\mathbf{q}}_n\right)$$

- Canonical rescaling

$$\hat{p}_n = \sqrt{\Lambda_n}\,\hat{\mathbf{p}}_n\,, \quad \hat{q}_n = \frac{1}{\sqrt{\Lambda_n}}\hat{\mathbf{q}}_n, \quad \Lambda_n^2 = -det(g_n)\,g_n^{00}\left(m^2 - \frac{2}{a^2}\,g_n^{11}\right)$$

- Canonical rescaling yields symmetric local terms
- Clear separation between local dynamics and coupling

Mapping to Gaussian operations

Structure after rescaling

$$H = \sum_n \left(\frac{\lambda_n}{2} (\hat{p}_n^2 + \hat{q}_n^2) + \kappa_{n,n+1}\, \hat{q}_n \hat{q}_{n+1} \right)$$

- Quadratic Hamiltonian $\Rightarrow$ Gaussian dynamics
- Lattice sites $\Rightarrow$ bosonic modes $(\hat{q}_n, \hat{p}_n)$
- Simulation using **Strawberry Fields**

Gaussian gates:

Displacement: $D(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$

Rotation: $R(\theta) = e^{i\theta(\hat{q}^2 + \hat{p}^2)/2}$

Coupling: $C_Z(s; q_n, q_m) = e^{is\, \hat{q}_n \hat{q}_m}$

Time evolution

Time Evolution (time independent Hamitonian)

$$U(t) = e^{-iHt}$$

- Hamiltonian $H = \sum_n H_n$
- Trotter–Suzuki approximation
  $U(t) \approx \left(\prod_n e^{-iH_n\delta t}\right)^{t/\delta t}$
- Decompose each step into Gaussian gates

$$U(t) = \left(\prod_{n=0}^{N-1} R\left(\frac{-\lambda_n}{2}\delta t\right) C_Z(\kappa_n \delta t)\ R\left(\frac{-\lambda_n}{2}\delta t\right)\right)^{t/\delta t}$$

Time Evolution (time dependent Hamitonian)

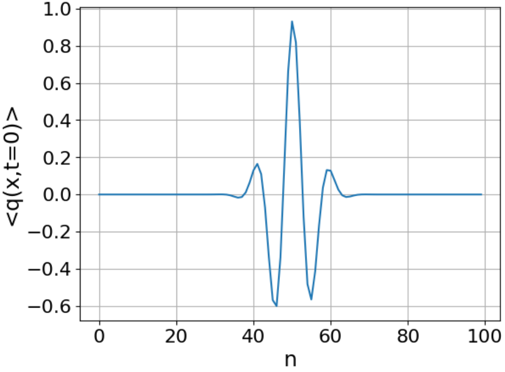
$$U(t, t_0) = \mathcal{T} e^{-i\int_{t_0}^{t} H(t')\,dt'}$$

- Discretize time $U(t) \approx \prod_{j=0}^{N_t-1} e^{-iH(t_j)\delta t}$
- Decompose each step into Gaussian gates (Trotter)

$$U(t) = \prod_{j}^{N_t-1}\left(\prod_{n}^{N-1} R\left(\frac{-\lambda_n}{2}\delta t\right) C_Z(\kappa_n \delta t)\ R\left(\frac{-\lambda_n}{2}\delta t\right)\right)$$

Strang splitting: $e^{A+B} \approx e^{A/2}e^{B}e^{A/2}$

Initial state



Gaussian wavepacket $\langle q(x,0)\rangle$

- Gaussian wavepacket

$$\phi(x) \sim e^{-\frac{(x-x_0)^2}{2\sigma^2}} e^{ik_0 x}$$

- Momentum from dispersion

$$\pi_k = -i\omega_k \phi_k$$

- Mapping to simulation

$$\alpha_n = \frac{1}{\sqrt{2}}(q_n + ip_n)$$

Results

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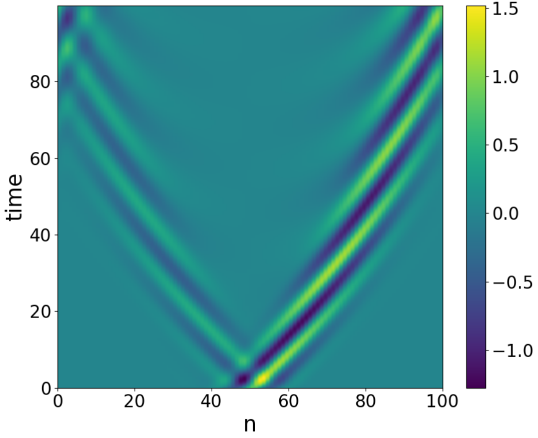
FLRW-metric

- Metric $\boxed{g_{\mu\nu}(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-a(t)^2}{1-kr^2} \end{pmatrix}}$
- Scale factor $a(t) = e^{Ht}$
- Simulation setup
  - time dependent metric
  - periodic spatial lattice
- Parameters
  - $H = 0.01$
  - $a_{lat} = 1$
  - $\delta t = 0.1$
  - $\boxed{k = 0}$



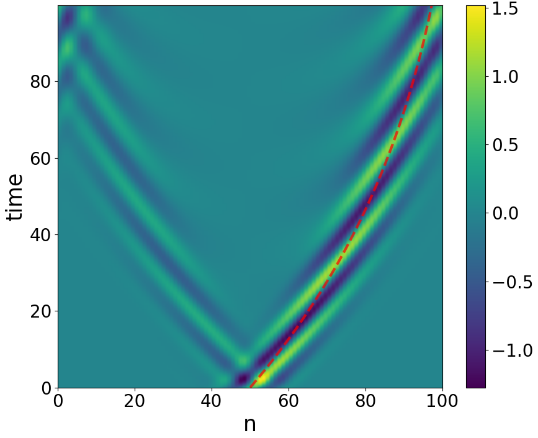
Field expectation value $\langle q(x,t) \rangle$ in FLRW space.

- Red-Line approximation:
  - Group velocity

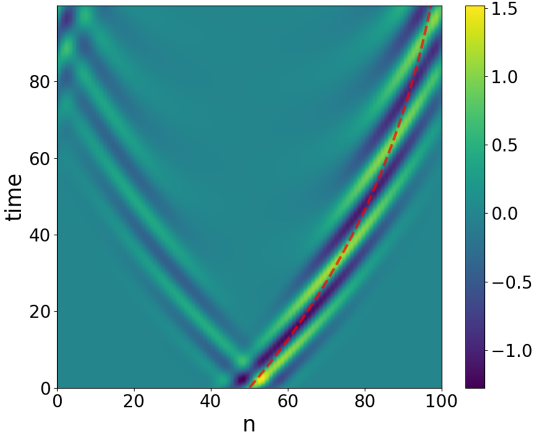
$$v_g(t) \approx \left. \frac{\partial \omega_k(t)}{\partial k} \right|_{k=k_0}$$

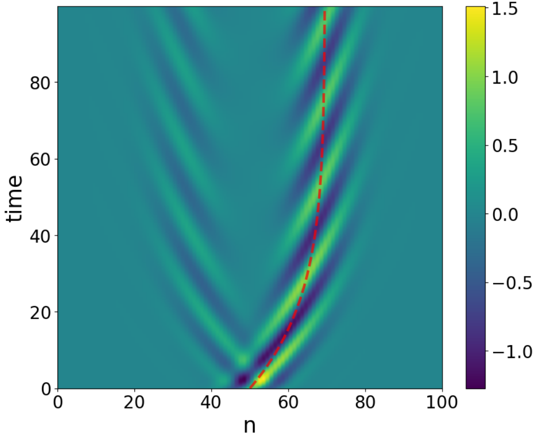
  - Trajectory of the wavepacket center

$$x(t_j) = x_0 + \sum_{l=0}^{j} v_g(t_l)\, \Delta t$$



Field expectation value $\langle q(x,t) \rangle$ in FLRW space.

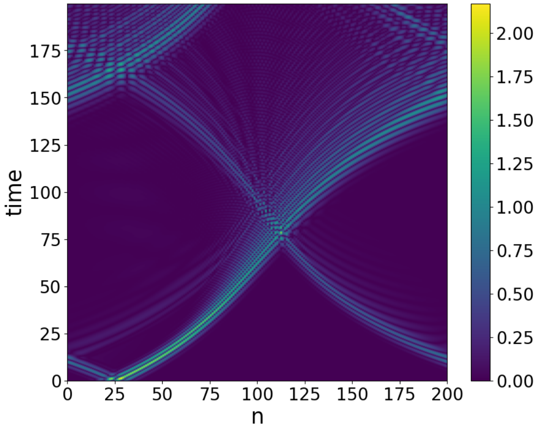




Field expectation values $\langle q(x,t)\rangle$ in FLRW space at $H \in [0.01, 0.03]$.

FLRW-metric

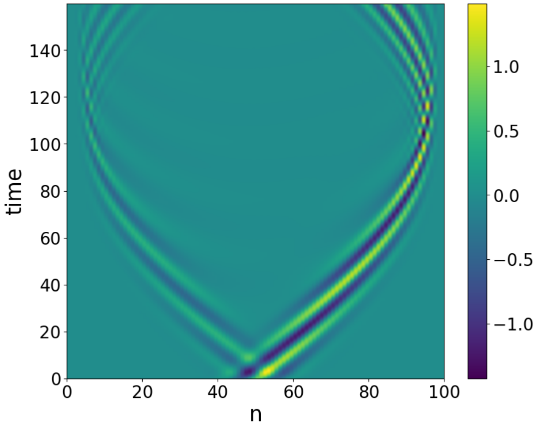
- Metric $$g_{\mu\nu}(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-a(t)^2}{1-kr^2} \end{pmatrix}$$
- Scale factor $a(t) = e^{Ht}$
- Simulation setup
  - time **and** space dependent metric
  - periodic spatial lattice
- Parameters
  - $H = 0.001$
  - $a_{lat} = 1$
  - $\delta t = 0.1$
  - $k = 0.01$



Absolute field expectation value $|\langle q(x,t)\rangle|$.

Schwarzschild-metric

- Metric $g_{\mu\nu}(t) = \begin{pmatrix} f(r) & 0 \\ 0 & -\frac{1}{f(r)} \end{pmatrix}$
- Scale factor $f(r) = 1 - \frac{r_s}{r}$
- Simulation setup
  - space dependent metric
  - periodic spatial lattice
- Parameters
  - $r_s = 2GM$
  - $a_{lat} = 1$
  - $\delta t = 0.1$
  - $G = 2$
  - $M = 4$



Field expectation value $\langle q(x,t)\rangle$ in Schwarzschild space.

Inspiration

- Steven Abel, Michael Spannowsky und Simon Williams (Juli 2025). “Real-time scattering processes with continuous-variable quantum computers”. In: *Physical Review A* 112.1
- Steven Abel, Michael Spannowsky und Simon Williams (Juli 2024). “Simulating quantum field theories on continuous-variable quantum computers”. In: *Physical Review A* 110.1

Theory

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- Robert M. Wald (1984). *General Relativity*. The University od Chicago Press
- Steven Weinberg (2005). *The Quantum Theory of Fields, Volume 1: Foundations*. Cambridge University Press