Universität Hamburg
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Comparing Gravitational Wave Spectra of Inflation Models

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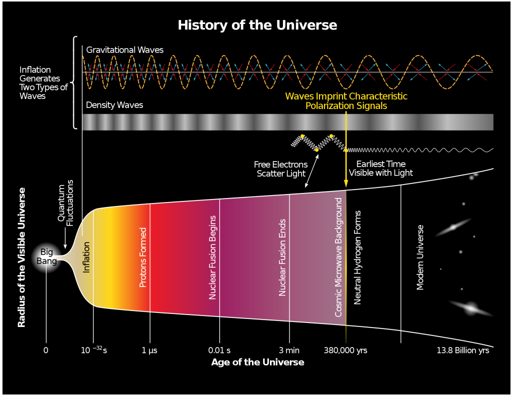
Agenda

- What is Inflation
- Gravitational Waves from Inflation
- Reheating
- Cosmolattice
- Models
- Results
- Appendix

What is Inflation?

- Proposed period of rapid expansion in the early universe
- Solves horizon and flatness problem
- Happened before the decoupling of the CMB
- Gravitational waves were produced → direct measurement
- Requires beyond standard model physics, usually a scalar field, inflaton field



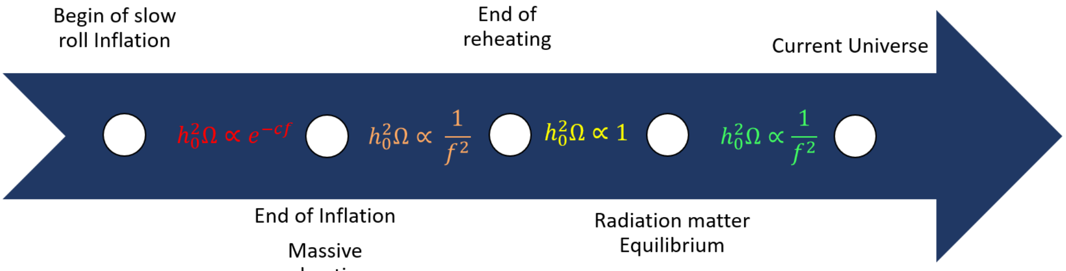
[Bogdan 2023]

Gravitational Waves from Inflation

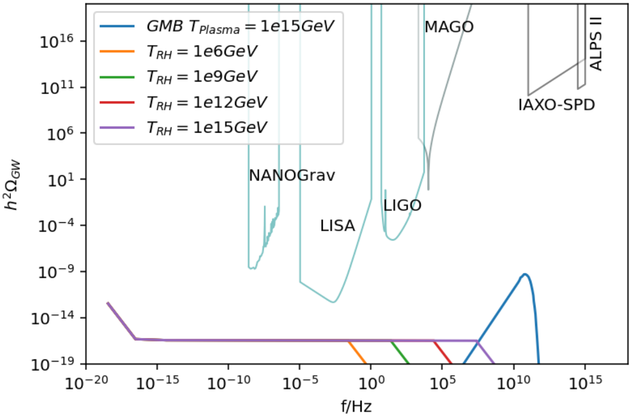
During Inflation quantum fluctuations get extended to classical scales $\rightarrow$ initial spectrum almost flat [Buchmuller et al. 2013]. The current spectrum depends mostly on [Maggiore 2018]:

- When does the mode enter the Hubble horizon (conserved outside, $\propto \frac{1}{a}$ within the horizon)
- How did the universe expand since then

This divides the spectrum into sections representing the different epochs in the history of the universe



Spectrum from inflation



[Ringwald, Schütte-Engel, and Tamarit 2021]

What happens after inflation

- After inflation the inflaton decays, in our case into another scalar field
- The universe becomes radiation dominated
- The potential is
$$V(\phi, \chi) = V(\phi) + \frac{1}{2} g^2 \phi^2 \chi^2$$
- Initially $\chi = 0$, g is small enough to be negligible during inflation
- After inflation ϕ and χ come to resonance
- Fluctuations of χ are increased exponentially $\rightarrow$ inhomogeneities $\rightarrow$ gravitational waves are produced
- Afterwards most energy is still in the inflaton and the rest is transferred through perturbative decays

Rest of the Reheating

If we assume a parabolic potential $V(\phi, \chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}g^2\phi^2\chi^2$,
ϕ follows

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + m^2\phi = 0$$

Here Γ is the decay rate, which becomes relevant around $H \approx \Gamma$.
Problem [Kofman, Linde, and Starobinsky 1997]:
Γ is proportional to the square of the oscillation amplitude $\Phi(t) \propto \frac{1}{t}$ and $H \propto \frac{1}{t}$
$\rightarrow$ perturbative decay never becomes dominant
$\rightarrow$ energy stays in the inflaton.
Solution: minimum of the inflaton is shifted away from 0

Rest of the Reheating

If $V(\phi)$ is a function of $(\phi - \sigma)$ we can define $\tilde{\phi} = \phi - \sigma$ and the potential becomes:

[Kofman, Linde, and Starobinsky 1997]

$$V(\tilde{\phi}, \chi) = \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{1}{2} g^2 \tilde{\phi}^2 \chi^2 + g^2 \sigma \tilde{\phi} \chi^2 + \frac{1}{2} g^2 \sigma^2 \chi^2$$

Now $\Gamma = \frac{g^4 \sigma^2}{8\pi m}$, and therefore the point $H \approx \Gamma$ is reached, but χ has an effective mass of $|g\sigma|$

$\rightarrow$ not radiation like

Solution: χ must decay into something radiation like

Simulating (P)reheating

- The preheating stage will be simulated with Cosmolattice [Figueroa et al. 2023]
- For the rest of reheating, we take the continuity equation, like [Antusch, Marschall, and Torrenti 2025]:

$$\dot{\rho} + 3H(P + \rho) = 0$$

$$\dot{\rho} + 3H\rho = -3HP$$

$$\text{for a massive field with } P = 0: \quad \dot{\rho} + (3H + \Gamma)\rho = 0$$

$$\text{and for radiation with } P = \frac{1}{3}\rho: \quad \dot{\rho} + (4H + \Gamma)\rho$$

- Why? In order To calculate the remnants of the spectrum, we use the formula $\Omega_{\text{GW current}}(f) = \frac{\rho_{c\text{ initial}}}{\rho_{c\text{ current}}}\left(\frac{a_{\text{initial}}}{a_{\text{current}}}\right)^4 \Omega(f)_{\text{GW initial}}$
- The amount of e-folds until the reheating point is needed, from then a temperature and the expansion till today expansion can be calculated

Simulating Reheating

In our case this leads to three equations:

$$\dot{\rho}_\phi + (3H + \Gamma_{\phi\chi})\rho_\phi = 0$$

$$\dot{\rho}_\chi + (3H + \Gamma_{\chi\gamma})\rho_\chi - \Gamma_{\phi\chi}\rho_\phi = 0$$

$$\dot{\rho}_\gamma + 4H\rho_\gamma - \Gamma_{\chi\gamma} = 0$$

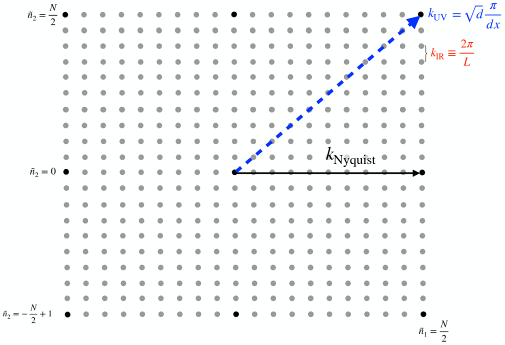
These will be solved in mathematica with $\Gamma_{\chi\gamma} = \beta \cdot \Gamma_{\phi\chi}$.

Critical assumptions:

- By the end of the lattice simulation, the fields have calmed down and are approximately homogeneous
- The interaction between the fields is negligible at the end of the lattice simulation
- Both fields can be regarded as separate massive fields
  $\implies$ the universe is matter dominated after preheating

Cosmolattice

- [Figueroa et al. 2023] Open source program for simulating scalar fields on a 3D lattice
- Oscillations, GW are simulated on the reciprocal lattice
  $\rightarrow$ size of the lattice limits spectral range
- Points p. dimension and minimal momentum are chosen appropriately
- Minimal cutoff is $k_{\mathrm{IR}} = \frac{2\pi}{N\delta x}$
- The maximum momentum is $k_{\mathrm{UV}} = \frac{\sqrt{3}N}{2} k_{\mathrm{IR}}$



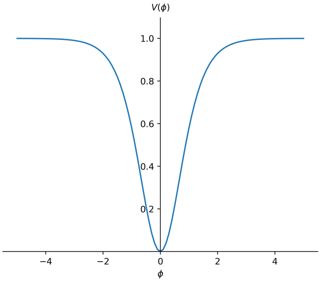
[Torrentí 2025]

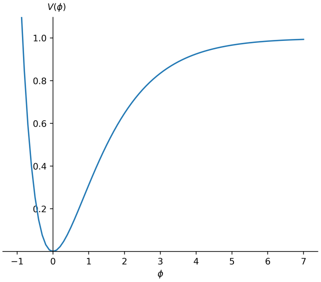
Inflation models

Two models were used

- T-Model α-attractor
- General shape of the potential is
  $V(\phi) = \frac{1}{2}\lambda^4 \tanh^2\left(\frac{\phi}{M\sqrt{6\alpha}}\right)$

- E-Model α-attractor
- General Shape of the potential is
  $V(\phi) = \frac{1}{2}\lambda^4\left(1 - e^{-\sqrt{\frac{3}{2\alpha}}\frac{\phi}{M}}\right)^2$





How can one derive these potentials?

Such potentials like this can be achieved:
models, that are non minimally coupled to the Ricci scalar in the Jordan frame.
Simulations are often done in the Einstein frame.
The Lagrangian in the Jordan frame for the E-model potential is :

[Kallosh, Linde, and Roest 2025]

$$\frac{1}{\sqrt{-g}}\mathcal{L}(\phi) = \frac{1}{2}(1+\zeta\phi)R - \frac{1}{2}g_{\mu\nu}\partial\phi_\mu\partial\phi_\nu - \frac{1}{2}m^2\phi^2$$

Reads in the Einstein frame, with $\tilde{g} = g \cdot (1+\zeta\phi)$and $\zeta = 1$:

$$\frac{1}{\sqrt{-\tilde{g}}}\tilde{\mathcal{L}}(\phi) = \frac{1}{2}\tilde{R} - \frac{1}{2}\tilde{g}_{\mu\nu}\mathcal{K}^2\partial_\mu\phi\partial_\nu\phi - \frac{1}{2(1+\phi)^2}m^2\phi^2$$

with $\mathcal{K}^2 = \frac{2.5+\phi}{(1+\phi)^2}$.

Recanonizing the kinetic term

Since

$$\mathcal{K} = \frac{\sqrt{2.5+\phi}}{(1+\phi)}$$

with a residuum at $\phi = -1$ of order 1.

Approximation of $\mathcal{K}$ with a Laurent expansion leads to

$$\frac{\sqrt{1.5}}{(1+x)}$$

Recanonanization the kinetic term gives us

$$\tilde{\phi} = \sqrt{1.5} \cdot \ln(\phi + 1)$$

the potential becomes

$$\tilde{V}(\tilde{\phi}) = \frac{1}{2(1+\phi)^2} m^2 \phi^2 = \frac{1}{2} m^2 \frac{(e^{\frac{1}{\sqrt{1.5}}\tilde{\phi})} + 1}{e^{\frac{2}{\sqrt{1.5}}\tilde{\phi}}} = \frac{1}{2} m^2 (e^{-\sqrt{\frac{2}{3}}\tilde{\phi}} - 1)^2$$

$$V(\tilde{\phi}) = \frac{1}{2}\lambda^4 \left(e^{-\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M}} - 1\right)^2$$

The factor $\frac{1}{M}$ comes when we define the initial potential as $V(\phi) = \frac{1}{2}m^2 M^2 \frac{\phi^2}{M^2}$.
$\implies \lambda^4 = m^2 M^2$ and $m = \frac{\lambda^2}{M}$
The parameter M does not depend on the model and simply serves for rescaling.

- A similar derivation can be made for the T-model potential
- For small field amplitudes Einstein and Jordan frame become interchangeable
  $\implies$ it becomes parabolic

Parameters for these models

For T-model potentials, the general formula is:

$$r = \frac{12\alpha}{N^2 + \frac{N}{2n}\sqrt{3\alpha(4n^2+3\alpha)} + \frac{3}{4}\alpha}$$

$$n_s = \frac{1 - \frac{2}{N} - \frac{3\alpha}{4N^2} + \frac{1}{2nN}(1-\frac{1}{N})\sqrt{3\alpha(4n^2+3\alpha)}}{1 + \frac{1}{2nN}\sqrt{3\alpha(4n^2+3\alpha)} + \frac{3\alpha}{4N^2}}$$

For $V \propto \tanh^{2n}(\frac{\phi}{\sqrt{6\alpha}})$.

Assuming $\alpha = \frac{1}{6}$ and $N = 60$ we get:

$n_s = 0.967$ $r = 0.000549$

[Kallosh, Linde, and Roest 2013][Kallosh, Linde, and Roest 2025]

For the E-model, the Formulas are:

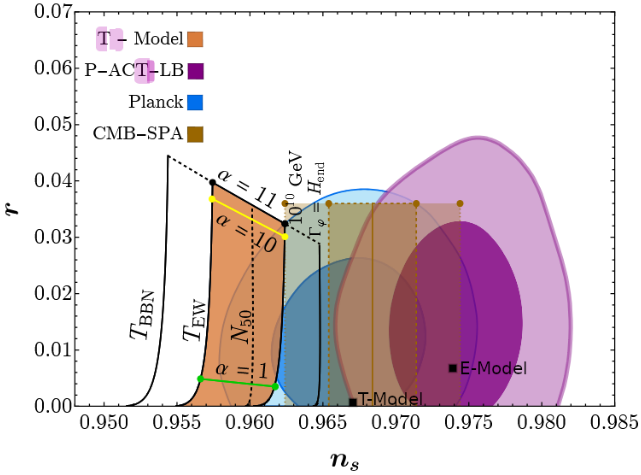
$$r \approx \frac{4}{N^{\frac{3}{2}}}$$

$$n_s \approx 1 - \frac{3}{2N}$$

Assuming $N = 60$:

$n_s = 0.974$ $r = 0.008607$

Where in the Parameter space are the models



[Ellis et al. 2025]

Other model Parameters

Goal is comparing the models, the same parameters and initial values were used for both The Model Parameters were:

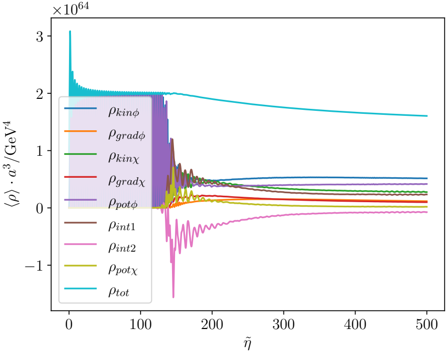
- $\lambda^4 = 9.72029 \cdot 10^{65}$
- $\phi_\text{I} = 3.39927 \cdot 10^{18}\text{GeV}$
- $\dot{\phi}_\text{I} = -2.25918 \cdot 10^{31}\text{GeV}^2$
- $M = 2.435 \cdot 10^{19}\text{GeV}$
- $\sigma = 2.345 \cdot 10^{16}$
- $\frac{g^2\sigma^2 M}{\lambda^2} = 0.5$
- $\chi_\text{I} = 0$
- $\dot{\chi}_\text{I} = 0$

The lattice parameters were:

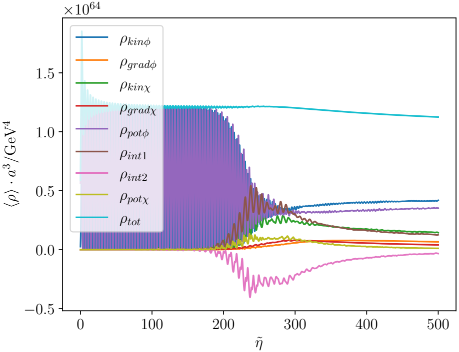
- $N = 440$
- $k_\text{IR} = 1$
- $\Delta t = 0.002$

Energy Components

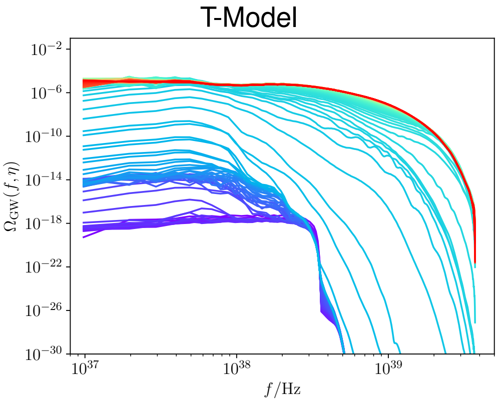
T-model

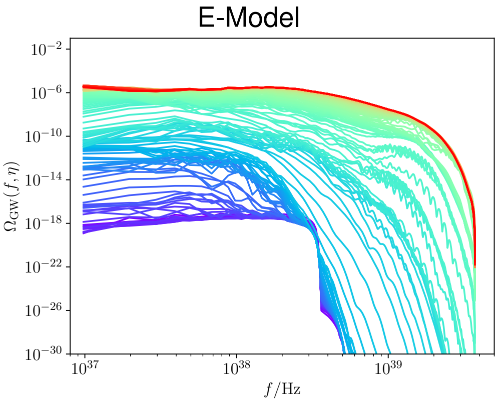


E-model



Spectrum


T-Model
10^{-2}
10^{-6}
10^{-10}
10^{-14}
10^{-18}
10^{-22}
10^{-26}
10^{-30}
$\Omega_{\rm GW}(f,\eta)$
10^{37}
10^{38}
10^{39}
$f/\rm Hz$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10^{-2}
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10^{-26}
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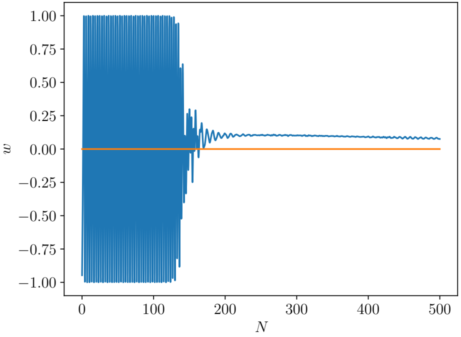
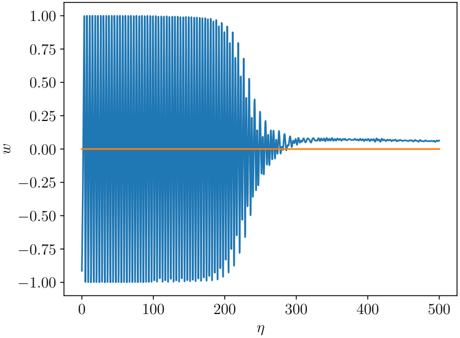
$\Omega_{\rm GW}(f,\eta)$
10^{37}
10^{38}
10^{39}
$f/\rm Hz$

Equation of state parameter

T-Model



E-Model



Underlying equations

$$\dot{\rho}_\phi + (3H + \Gamma_{\phi\chi})\rho_\phi = 0$$

$$\dot{\rho}_\chi + (3H + \beta\Gamma_{\phi\chi})\rho_\chi - \Gamma_{\phi\chi}\rho_\phi = 0$$

$$\dot{\rho}_\gamma + 4H\rho_\gamma - \beta\Gamma_{\phi\chi} = 0$$

While $\Gamma_{\phi\chi} = \frac{g^4\sigma^2 M}{8\pi\lambda^2}$, β is independent of lattice parameters, will be tested for a range
Assumption check:

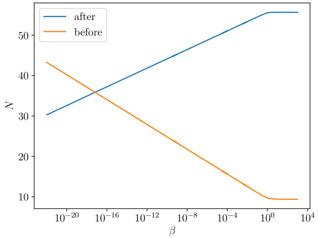
- The interaction terms were added separately (both, none, only one)
- This did not have an effect

E-folds since the creation of the spectrum

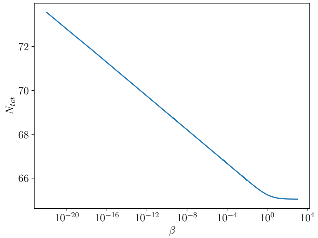
- For low β: Many of e-folds till reheating point,
  amount since reheating point smaller but not below 24.2
- For high β: The amount of e-fold till reheating point goes down to ≈ 10, the amount after saturates at $\approx 55 \rightarrow$ overall fewer

For the T-model

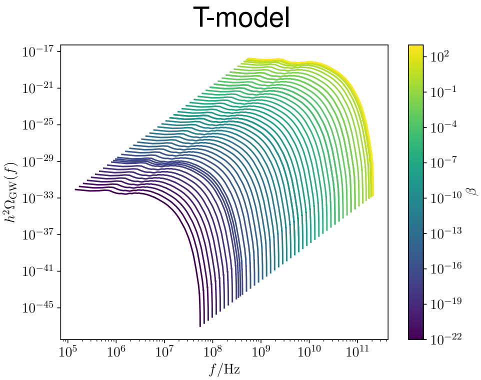
e-folds before and after rd start

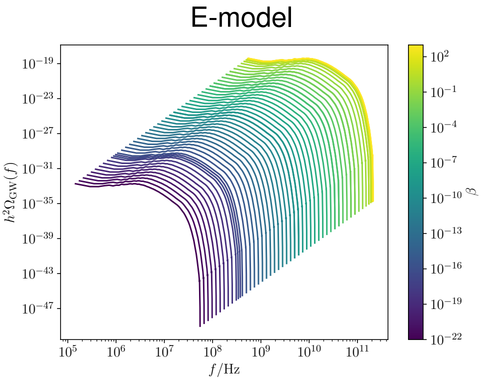


total amount of e-folds

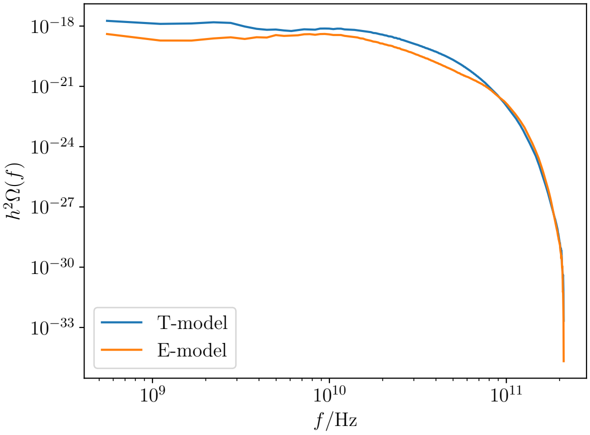


Remnant spectra for different ratios

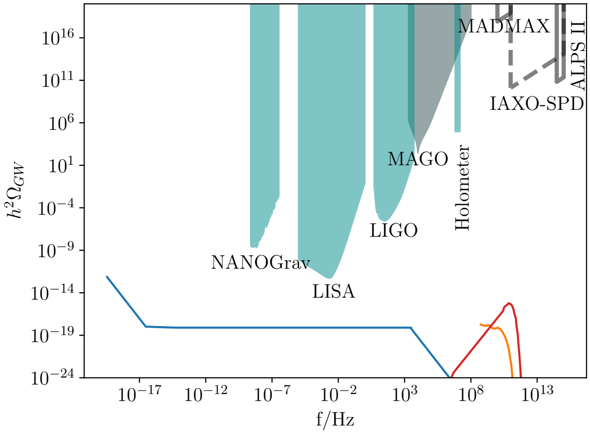




Comparison of the two spectra



Spectrum of inflation and reheating for T-model



Conclusion

- Comparison not possible since these models are too similar, though probing might be possible
- Spectrum might collide with CGMB from [Ringwald, Schütte-Engel, and Tamarit 2021], cutoff would not be visible
- Strong spectrum for the frequency range
- Further adjustment to parameters might still result in something stronger
  $\rightarrow$ e.g. for increasing the value of $\Gamma_{\phi\chi}$ by 10^5 while leaving the rest as is, I got $h^2\Omega(f) \approx 10^{-10}$
- Achievable by increasing coupling g
- Still needs to be integrated, properly fitted
  $\rightarrow$ e.g. [Ellis et al. 2025] the inflaton mass must be lowered to fit the scalar amplitude

Thank you for your attention

Questions?

Sources

- Antusch, Stefan, Daniel G. Figueroa, et al. (Feb. 2022). “Characterizing the post-inflationary reheating history, Part I: single daughter field with quadratic-quadratic interaction”. In: *Physical Review D* 105.4. arXiv:2112.11280 [astro-ph], p. 043532. ISSN: 2470-0010, 2470-0029. DOI: 10.1103/PhysRevD.105.043532. URL: http://arxiv.org/abs/2112.11280 (visited on 09/23/2025).
- Antusch, Stefan, Kenneth Marschall, and Francisco Torrenti (July 2025). *Equation of state during (p)reheating with trilinear interactions*. arXiv:2507.13465 [astro-ph]. DOI: 10.48550/arXiv.2507.13465. URL: http://arxiv.org/abs/2507.13465 (visited on 11/04/2025).
- Bogdan, Dennis (2023). In: URL: https://en.wikipedia.org/wiki/Cosmic_inflation#/media/File:History_of_the_Universe.svg.

Sources

- Buchmuller, Wilfried et al. (Oct. 4, 2013). *The Gravitational Wave Spectrum from Cosmological B-L Breaking*. DOI: 10.48550/arXiv.1305.3392. arXiv: 1305.3392. URL: http://arxiv.org/abs/1305.3392 (visited on 11/28/2024).
- Ellis, John et al. (Oct. 2025). *Constraints on Attractor Models of Inflation and Reheating from Planck, BICEP/Keck, ACT DR6, and SPT-3G Data*. en. arXiv:2510.18656 [hep-ph]. DOI: 10.48550/arXiv.2510.18656. URL: http://arxiv.org/abs/2510.18656 (visited on 11/12/2025).
- Figueroa, Daniel G. et al. (Feb. 2023). "CosmoLattice". In: *Computer Physics Communications* 283, p. 108586. ISSN: 00104655. DOI: 10.1016/j.cpc.2022.108586. arXiv: 2102.01031 [astro-ph]. URL: http://arxiv.org/abs/2102.01031 (visited on 04/24/2025).

Sources

Kallosh, Renata, Andrei Linde, and Diederik Roest (Nov. 2013). “Superconformal Inflationary α-Attractors”. In: *Journal of High Energy Physics* 2013.11. arXiv:1311.0472 [hep-th], p. 198. ISSN: 1029-8479. DOI: 10.1007/JHEP11(2013)198. URL: http://arxiv.org/abs/1311.0472 (visited on 04/29/2025).

— (Apr. 3, 2025). *A simple scenario for the last ACT*. DOI: 10.48550/arXiv.2503.21030. arXiv: 2503.21030[hep-th]. URL: http://arxiv.org/abs/2503.21030 (visited on 05/07/2025).

Kofman, Lev, Andrei Linde, and Alexei Starobinsky (Sept. 1997). “Towards the Theory of Reheating After Inflation”. en. In: *Physical Review D* 56.6. arXiv:hep-ph/9704452, pp. 3258–3295. ISSN: 0556-2821, 1089-4918. DOI: 10.1103/PhysRevD.56.3258. URL: http://arxiv.org/abs/hep-ph/9704452 (visited on 08/01/2025).

Sources

- Maggiore, Michele (2018). *Gravitational waves Volume 2: astrophysics and cosmology*. Oxford University Press. ISBN: 978-0-19-857089-9.
- Ringwald, Andreas, Jan Schütte-Engel, and Carlos Tamarit (Mar. 1, 2021). "Gravitational Waves as a Big Bang Thermometer". In: *Journal of Cosmology and Astroparticle Physics* 2021.3, p. 054. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2021/03/054. arXiv: 2011.04731 [hep-ph]. URL: http://arxiv.org/abs/2011.04731 (visited on 11/13/2024).
- Torrentí, Francisco (2025). "Lecture 1: Welcome to the Lattice!" In: URL: https://indico.ific.uv.es/event/8110/contributions/27274/attachments/13452/19917/CLschool25-Lecture1_Torrenti.pdf.

Gravitational waves from inflation

Main Production Mechanism

During Inflation Quantum fluctuations get extended to classical scales. This is the main mechanism and gives rise to an almost flat spectrum [Buchmuller et al. 2013]

The plane wave equation in the TT Gauge is

$$\ddot{h}_{\mu\nu} - \Delta h_{\mu\nu} = 16\pi T_{\mu\nu}$$

In comoving coordinates and conformal time that becomes

$$h''_{\mu\nu} + 2\frac{a'}{a}h'_{\mu\nu} - \Delta h_{\mu\nu} = 16\pi G T_{\mu\nu}$$

here $a' = \frac{da}{d\tau}$ with $a d\tau = dt$

Gravitational waves from inflation

$$h''_{\mu\nu} + 2\frac{a'}{a}h'_{\mu\nu} - \Delta h_{\mu\nu} = 16\pi G T_{\mu\nu}$$

Which written in terms of the Fourier transform with $\tilde{h}_{\mu\nu}$ and in the absence of a source is

$$\tilde{h}''_{\mu\nu} + (k^2 - \frac{a''}{a})\tilde{h}_{\mu\nu} = 0$$

using $\frac{\ddot{a}}{a} = H^2 + \dot{H}$ and $\dot{H} \approx 0$ we can write

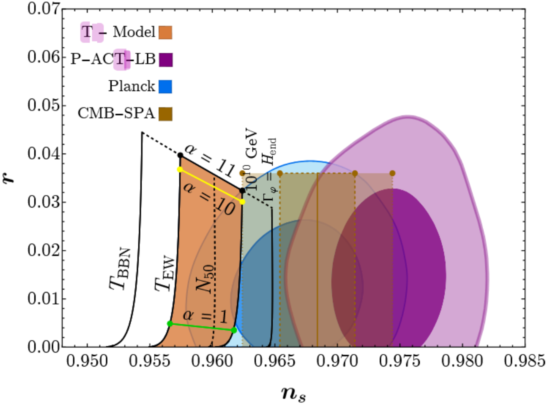
$$\frac{a''}{a} = (aH^2) = \frac{1}{r_{\text{CM}}}$$

Super- and Sub-horizon modes

- **Subhorizon modes**
  - $k \gg aH$
  - $$\tilde{h}''_{\mu\nu} + k^2 \tilde{h}_{\mu\nu} = 0$$
    $$\implies \tilde{h}_{\mu\nu} \propto \cos(k\tau)$$
    $$\implies h_{\mu\nu} \propto \frac{1}{a}\cos(k\tau)$$
  - Suppressed as a grows
- **Superhorizon modes**
  - $k \ll aH$
  - $$\tilde{h}''_{\mu\nu} - \frac{a''}{a}\tilde{h}_{\mu\nu} = 0$$
    $$\implies a h''_{\mu\nu} + 2a' h'_{\mu\nu} = 0$$
    $$\implies h_{\mu\nu} = A + B\int_{\tau_I}^{\tau_F} \frac{d\tau}{a(\tau)^2}$$
  - has a conserved component

Since modes outside the horizon are conserved and modes inside the horizon are damped, the relevant question becomes when does a mode enter the horizon

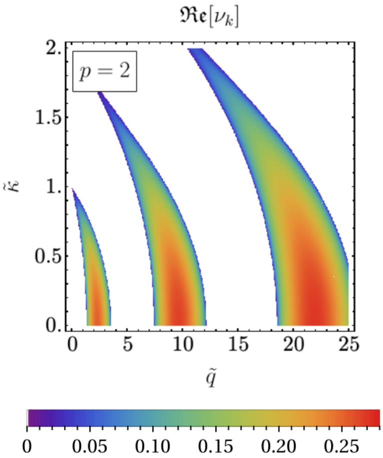
Parameters



[Ellis et al. 2025]

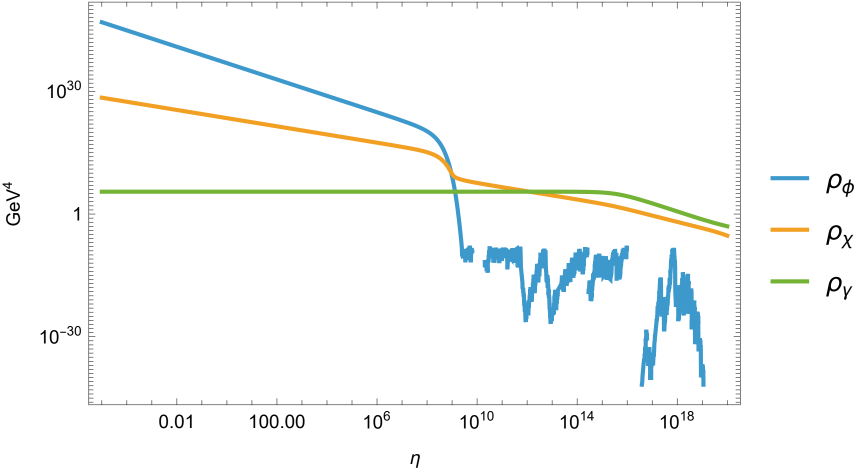
Model Parameters

The reason to define $\frac{g^2\sigma^2 M}{\lambda^2}$ is, that Cosmolattice normalizes the potential to the effective inflaton mass $\frac{\lambda^2}{M}$. Also, the effective mass ratio can be used to determine the resonance modes.
Defining the resonance parameter
$q = \frac{g^2\sigma^2 M}{\lambda^2}$



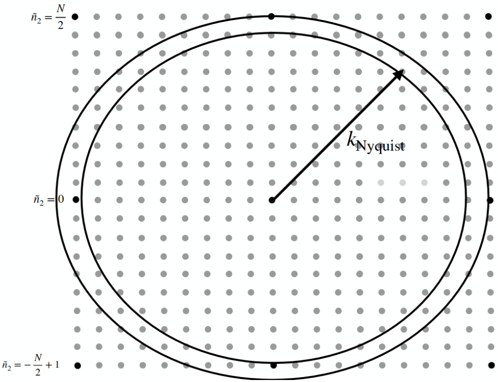
[Antusch, Figueroa, et al. 2022]

Rest of reheating



Gravitational Waves in Cosmolattice

- The source term are the field oscillations projected by the anisotropic stress tensor
- The modes are averaged for multiple points over bins
- The different bins are made of spherical shells on the reciprocal lattice $\longrightarrow$ not all bins have the same amount of points.



[Torrentí 2025]

Mistake in the parameter

- In the thesis I wrote $\Gamma_{\phi\chi} = \frac{g^4\sigma^2}{8\pi m_\phi^2}$
- $\Gamma_{\phi\chi} = \frac{g^4\sigma^2}{8\pi m_\phi}$ $\rightarrow$ a factor of an inflaton mass larger
- had around $59 - 60$ e-folds since creation, with the correction it's $65 - 70$

Low frequency simulation