Instanton corrected HK metrics

Warning: rough notes, may be innacurate/incomplete at some points

1 Motivation

We consider a 4d $\mathcal{N}=2$ gauge theory. Given such a theory, one can perform dimensional reduction on S^1 and obtain a 3d $\mathcal{N}=4$ theory, which turns out to be a (SUSY) σ-model. In particular, the scalar bosonic fields give a map of the form

$$\phi : \mathbb{R}^{2,1} \to (\mathcal{M}, g) \tag{1.1}$$

where $(\mathcal{M}, g)$ is the moduli space of vacua of the 3d theory. When we restrict to the special case of theories of class S, then $\mathcal{M}$ matches the moduli space of flat connections from previous talks.

$\mathcal{N}=4$ SUSY in $3d$ implies strong restrictions on the geometry of $\mathcal{M}$, it implies that $\mathcal{M}$ is hyperkähler (HK) (see Gaumé-Friedman 81, or Hitchin, Karlhede, Lindström, Roček 87). These turn out to be quite special manifolds. If you are a physicist, then fully describing the resulting HK manifold is important for fully describing the low energy EFT; while if you are mathematician, you can think of the previous procedure as a factory for HK manifolds.

We will start by explaining what HK manifolds are, and then presenting the plan for the rest of the talk.

Let us recall the corresponding definitions. Recall that a Kähler manifold is a tuple (M, g, I) such that

- (M, g) is Riemannian.
- (M, I) is complex.
- $I \in O(TM, g)$ and $\nabla I = 0$.

For this class of manifolds, it automatically follows that $\omega(-,-) = g(I-,-)$ is a symplectic form, usually called the Kähler form.

On the other hand, a HK manifold is a tuple (M, g, I_1, I_2, I_3) such that

- Each (M, g, I_i) is Kähler.
- (I_1, I_2, I_3) satisfy the imaginary quaternion relations.

We in particular have three associated Kähler forms $\omega_i(-,-) = g(I_i-,-)$. In fact, for any $a \in S^2 \subset \mathbb{R}^3$, we can form

$$I_a = a^i I_i\,, \quad \omega_a = a^i \omega_i\,. \tag{1.2}$$

Then (M, I_a, ω_a) is again Kähler. Even more, parametrizing $S^2 \cong \mathbb{C}P^1$ by a holomorphic linear coordinate ζ, the combination

$$\varpi(\zeta) = -\frac{i}{2}\zeta^{-1}(\omega_1 + i\omega_2) + \omega_3 - \frac{i}{2}\zeta(\omega_1 - i\omega_2) \tag{1.3}$$

is such that $(M, I_\zeta, \varpi(\zeta))$ is a holomorphic symplectic manifold. From the family $\varpi(\zeta)$ one can recover ω_i, $i = 1, 2, 3$, and from these, one can reconstruct the HK structure.

HK manifolds are in particular Calabi-Yau. Some examples include $\mathbb{H}^n$, $(T^4)^n$, $K3$, T^*M when M is affine special Kähler, moduli of Higgs bundles/flat connections, etc.

The description of the HK structure obtained by the S^1 dimensional reduction was done in (Gaiotto-Moore-Neitzke 09). Their description is done via twistorial methods, in terms of a family ζ-family of Darboux coordinates for $\varpi(\zeta)$ than can be thought as "cluster-like" coordinates. In particular, the family of coordinates is labeled by $\mathcal{X}_\gamma(-,\zeta)$, $\zeta \in \mathbb{C}^\times$, and they give a family of Darboux coordinates for $\varpi(\zeta)$. These coordinates in turn are found by solving TBA-like integral equations. Because of this, it is not obvious that they match the cluster-like coordinates built for the theories of class S of the last talk, but (under certain conditions), it is argued by Gaiotto-Moore-Neitzke that they do.

For the talk, we have the following goals:

- First, we would like to present the main ideas of the GMN construction. In particular, the appearance of the cluster-like coordinates describing the HK structure are solving a system of TBA-like equations.
- Finally, see what the construction reduces to when restricting to a theory of class S from the last talk.

2 Setting and brief summary of GMN's construction

In the following, we will give a rough idea of what goes into the GMN construction. We will first define a tuple $(\mathcal{B}, D, \Gamma, Z, \Omega)$ associated to the 4d $\mathcal{N}=2$ theory. We explain how from $(\mathcal{B}, D, \Gamma, Z)$ one can construct a preliminary and explicit "semi-flat" HK structure. And then how to get the full HK structure by "correcting" the semi-flat structure with the Ω data.

GMN's construction uses a tuple of data that one obtains from the $4d$ $\mathcal{N}=2$ theory. This tuple, in turn, can be formulated independently of giving a such a theory. This is a tuple $(\mathcal{B}, D, \Gamma, Z, \Omega)$, where:

- (Coulomb Branch) $\mathcal{B}$ is a $\mathbb{C}$-manifold. We denote $\dim_{\mathbb{C}}(\mathcal{B}) = n$.
- (Singular locus) $D \subset \mathcal{B}$ is a divisor. We denote $\mathcal{B}' = \mathcal{B} - D$.
- (Charge lattice) $\Gamma \to \mathcal{B}'$ is local system of rank $2n$, with a symplectic pairing $\langle -,- \rangle : \Gamma \times \Gamma \to \mathbb{Z}$. We assume that $\langle -,- \rangle$ admits Darboux frames $(\widetilde{\gamma}_i, \gamma^i)$. Sometimes, one needs to extend the charge lattice by a "flavor lattice" Γ^f, such that Γ fits into
$$0 \to \Gamma^f \to \Gamma \to \Gamma^g \to 0\,, \tag{2.1}$$
Here Γ^f is a trivial local system, and Γ^g is the rank $2n$ lattice with the symplectic pairing from before. In the following, we will assume that $\Gamma^f = \{0\}$ for simplicity. If you are interested in the details, see (Neitzke 13).
- (Central charge) Z is a holomorphic section of $\Gamma^* \otimes \mathbb{C} \to \mathcal{B}'$. If γ is a local section of Γ, then Z_γ is a holomorphic function.
- (BPS indices) $\Omega : \Gamma - \{0\} \to \mathbb{Z}$ is a function of sets. It satisfies the Kontsevich-Soibelman WCF and $\Omega(\gamma) = \Omega(-\gamma)$. In particular $\Omega(\gamma)$ is locally constant away from a real codim 1 "wall" $\mathcal{W} \subset \mathcal{B}'$.

Furthermore, the tuple $(\mathcal{B}', \Gamma, Z)$ satisfies further conditions that guarantee that $\mathcal{B}'$ carries an affine special Kähler structure (ASK). Namely:

- We assume that given $(\widetilde{\gamma}_i, \gamma^i)$, $\{Z_{\gamma^i}\}$ and $\{Z_{\widetilde{\gamma}_i}\}$ define holomorphic coordinates on $\mathcal{B}'$.
- The central charge satisfies
$$\langle dZ \wedge dZ \rangle = 0\,. \tag{2.2}$$
This implies that if $dZ_{\widetilde{\gamma}_i} = \tau_{ij} dZ_{\gamma^j}$, then $\tau_{ij} = \tau_{ji}$.

- $\omega \in \Omega^2(\mathcal{B}')$ defined by

$$\omega := \langle dZ \wedge d\overline{Z} \rangle = \mathrm{Im}(\tau_{ij}) dZ_{\gamma^i} \wedge d\overline{Z}_{\gamma^j} \tag{2.3}$$

  defines a Kähler structure on $\mathcal{B}'$.

In particular, $\tau_{ij} = \tau_{ji}$ implies that the holomorphic one form $Z_{\widetilde{\gamma}_i} dZ_{\gamma^i}$ is closed, so there is a (local) holomorphic function $\mathfrak{F}(Z_{\gamma^i})$ such that $d\mathfrak{F} = Z_{\widetilde{\gamma}_i} dZ_{\gamma^i}$. I.e.

$$\frac{\partial \mathfrak{F}}{\partial Z_{\gamma^i}} = Z_{\widetilde{\gamma}_i}, \quad \tau_{ij} = \frac{\partial^2 \mathfrak{F}}{\partial Z_{\gamma^i} \partial Z_{\gamma^j}} \,. \tag{2.4}$$

Hence, $\{Z_{\widetilde{\gamma}_i}, Z_{\gamma^i}\}$ form a conjugate system of special holomorphic coordinates for each choice of Darboux frame of Γ. From just the tuple $(\mathcal{B}', \Gamma, Z)$ and the associated ASK geometry, one can define a HK structure over the torus fibration $\mathcal{M}' \to \mathcal{B}'$, defined by

$$\mathcal{M}'|_p := \{\theta : \Gamma_p \to \mathbb{R}/2\pi\mathbb{Z} \mid \theta_\gamma + \theta_{\gamma'} = \theta_{\gamma+\gamma'} + \pi\langle \gamma, \gamma' \rangle\} \,. \tag{2.5}$$

The HK structure on $\mathcal{M}' \subset \mathcal{M}$ is given by the rigid c-map construction, and its sometimes called the semi-flat metric. It is given by

$$\begin{aligned}
g^{\mathrm{sf}} &:= R\mathrm{Im}(\tau_{ij}) dZ^i d\overline{Z}^j + \frac{\mathrm{Im}(\tau)^{ij}}{4\pi^2 R} W_i \overline{W}_j, \quad W_i := d\theta_{\widetilde{\gamma}_i} - \tau_{ij} d\theta_{\gamma^j} \,, \\
\omega_1^{\mathrm{sf}} + i\omega_2^{\mathrm{sf}} &:= -\frac{1}{2\pi} \langle dZ \wedge d\theta \rangle \,, \\
\omega_3^{\mathrm{sf}} &:= \frac{R}{4} \langle dZ \wedge d\overline{Z} \rangle - \frac{1}{8\pi^2 R} \langle d\theta \wedge d\theta \rangle \,.
\end{aligned} \tag{2.6}$$

It is called semi-flat because it restricts to a flat metric on the fibers $\mathcal{M}'|_p \cong (S^1)^{2n}$ of $\mathcal{M}' \to \mathcal{B}'$. The parameter R above equals the radius of the compactification circle.

Furthermore, if $\zeta \in \mathbb{C}P^1$ parametrizes the complex structures of the HK structure, then a holomorphic symplectic form in holomorphic structure I_ζ is given for $\zeta \in \mathbb{C}^\times$ by

$$\varpi^{\mathrm{sf}}(\zeta) := -\frac{i}{2}(\omega_1^{\mathrm{sf}} + i\omega_2^{\mathrm{sf}}) + \omega_3^{\mathrm{sf}} - \frac{i}{2}(\omega_1^{\mathrm{sf}} - i\omega_2^{\mathrm{sf}}) \,. \tag{2.7}$$

This family can in turn be written as

$$\varpi^{\mathrm{sf}}(\zeta) := \frac{1}{8\pi^2 R} \langle d\log(\mathcal{X}^{\mathrm{sf}}(\zeta)) \wedge d\log(\mathcal{X}^{\mathrm{sf}}(\zeta)) \rangle, \quad \mathcal{X}_\gamma^{\mathrm{sf}}(\theta, \zeta) := \exp(\pi\zeta^{-1} R Z_\gamma + i\theta_\gamma + \pi\zeta R \overline{Z}_\gamma) \,, \tag{2.8}$$

so that, given a local Darboux frame $(\widetilde{\gamma}_i, \gamma^i)$ of Γ, $(\log(\mathcal{X}_{\widetilde{\gamma}_i}^{\mathrm{sf}}(\zeta)), \log(\mathcal{X}_{\gamma^i}^{\mathrm{sf}}(\zeta)))$ give holomorphic Darboux coordinates for $\varpi^{\mathrm{sf}}(\zeta)$.

So far, we have only dealt with the data $(\mathcal{B}, D, \Gamma, Z)$. The remaining data of the BPS indices are used to construct the instanton corrections as follows. The idea is to define new coordinates

$$\mathcal{X}_\gamma(\theta, \zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(\theta, \zeta) \mathcal{X}_\gamma^{\mathrm{inst}}(\theta, \zeta) \tag{2.9}$$

such that

$$\varpi(\zeta) := \frac{1}{8\pi^2 R} \langle d\log(\mathcal{X}(\zeta)) \wedge d\log(\mathcal{X}(\zeta)) \rangle \tag{2.10}$$

defines a twistor family of holomorphic symplectic forms associated to a new HK structure on $\mathcal{M}'$. Physically, this new HK structure captures non-perturbative quantum corrections of the 3d EFT, coming from particles wrapping around the compactification circle S^1.

In the following, we will present what the new coordinates are (without much motivation) and then do some comments about them. The new $\mathcal{X}_\gamma$ are found by solving a system of TBA-like integral equations given by

$$\mathcal{X}_\gamma(\theta,\zeta) = \mathcal{X}^{\text{sf}}_\gamma(\theta,\zeta)\exp\Big[-\frac{1}{4\pi i}\sum_{\gamma'\in\Gamma_{\pi(\theta)}}\Omega(\gamma')\langle\gamma,\gamma'\rangle\int_{\mathbb{R}_-Z_{\gamma'}}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log(1-\mathcal{X}_{\gamma'}(\theta,\zeta'))\Big]. \tag{2.11}$$

First, notice that the coordinates are actually discontinuous in the ζ variable along the rays $\mathbb{R}_-Z_\gamma$ with $\gamma\in\text{Supp}(\Omega)$. These are the so called BPS rays. The discontinuity is computed via usual contour integral methods, giving a jump for $\mathcal{X}_\gamma$ along $\mathbb{R}_-Z_{\gamma'}$ of the form

$$\mathcal{X}_\gamma \to \mathcal{X}_\gamma \prod_{\widetilde{\gamma}:Z_{\widetilde{\gamma}}\in\mathbb{R}_-Z_{\gamma'}}(1-\mathcal{X}_{\widetilde{\gamma}})^{\Omega(\widetilde{\gamma})\langle\widetilde{\gamma},\gamma\rangle}. \tag{2.12}$$

Furthermore, we note that $\varpi(\zeta)$ is actually invariant under these jumps, so in particular $\varpi(\zeta)$ is continous (in fact holomorphic) in $\zeta\in\mathbb{C}^\times$, even though the coordinates are not.

On the other hand, for fixed $\theta\in\mathcal{M}'$, these coordinates are argued to be uniquely characterized by such discontinuities, plus certain asymptotic conditions as $\zeta\to 0,\infty$ or $R\to\infty$.

However, there is another issue that must be solved. As we move over the base $\pi(\theta)=u$, the numbers $\Omega(\gamma,u)$ jump along a real codimension 1-wall $\mathcal{W}$. This might in principle lead you to think that the resulting $\varpi(\zeta)$ is discontinuous along $\pi^{-1}(\mathcal{W})$. The wall-crossing formula implies that at $u\in\mathcal{W}$, the overall discontinuity of $\mathcal{X}_\gamma$ (in the ζ-variable) is independent on which side we approach the wall. By the unique characterization of the coordinates $\mathcal{X}_\gamma$ by their jumps in the ζ-variable and their asymptotics, we conclude that they are continuous over the wall $\mathcal{W}$.

What about the solutions to the TBA-equations? In certain situations, for fixed $(\theta,\zeta)\in\mathcal{M}'\times\mathbb{C}^\times$ one can in principle find a solution to such equations by iteration. However, the mathematical details of the resulting domain of definition and signature of the resulting metric have (as far as I know) in general not been worked out. For physically realized BPS spectrums, the HK structure is expected to be positive definite and not only exist on $\mathcal{M}'$ but to also extend (with possibly mild singularities) over $D\subset\mathcal{B}$ to a space $\mathcal{M}\to\mathcal{B}$.

2.1 The case of theories of class S

We will now state what $(\mathcal{B},D,\Gamma,Z,\Omega)$ is for the theories of class S, where $\mathcal{M}$ is expected to match $\mathcal{M}_{\text{Higgs}}=\mathcal{M}_\flat$. For simplicity, we will restrict to the case where the group associated to the theory is $SU(2)$, which corresponds to considering $SU(2)$-harmonic bundles (or $SL(2,\mathbb{C})$-Higgs bundles, or $SL(2,\mathbb{C})$-flat connections). We fix a Riemann surface $\overline{\Sigma}$ and a finite set of points $P=\{P_i\}\subset\overline{\Sigma}$. We furthermore set $\Sigma=\overline{\Sigma}-P$ and to each puncture P_i choose $m_i\in\mathbb{C}$, $m_i^{\mathbb{R}}\in\mathbb{R}/2\pi\mathbb{Z}$. Then:

- $\mathcal{B}$ is the set of quadratic differentials ϕ with double pole at P_i and residue m_i^2.
- $D\subset\mathcal{B}$ is the locus of ϕ's having at least one non-simple zero.
- $\Gamma_\phi=H_1(\Sigma_\phi,\mathbb{Z})_{\text{odd}}$, where $\Sigma_\phi\subset T^*\Sigma$ is the corresponding spectral curve of ϕ and odd denotes the subgroup invariant under the obvious involution. The pairing is given by the intersection pairing.
- Z is given by

$$Z_\gamma := \int_\gamma \lambda \tag{2.13}$$

  where λ is the Lioville 1-form on $T^*\Sigma$.

- The $\Omega(u,\gamma)$ are computed via the techniques of the previous talk + KSWCF.

Furthermore, if we denote by $\mathcal{X}^{\text{TBA}}_\gamma(x,\zeta)$ the coordinates describing the HK structure and by $\mathcal{X}^\theta_\gamma(x,\zeta)$ the coordinates from the previous talk, the GMN argue that (at least for $R>>0$), we should have

$$\mathcal{X}^{\text{TBA}}_\gamma(x,\zeta)=\mathcal{X}^{\text{Arg}(\zeta)}_\gamma(x,\zeta) \tag{2.14}$$