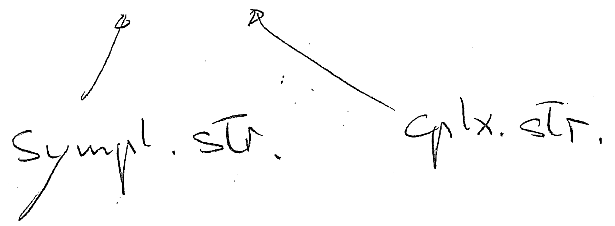
Mirror symmetry and cluster algebras
– an informal introduction –

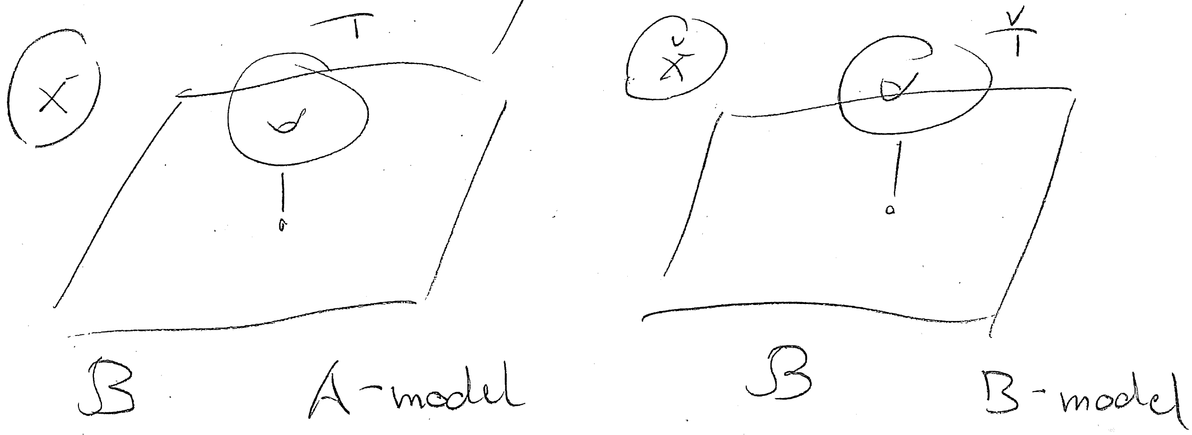
(1) A geometric constr. motivated by string th.

Conjecture: CY-mfds. come in pairs:

$$(X, \omega, I) \longleftrightarrow (\check{X}, \check{\omega}, \check{I})$$



SYZ: Duality of torus fibrations



Physics motivation: $X, \check{X}$ ~ moduli sp. of D-branes

Why? In B-model ∃ D0-branes (~ skyscraper sheaves, objects of $D^b(\check{X})$) pointlike localized ⇝ probe (cplx.) geom. of $\check{X}$: Mod. sp. of D0 ≡ $\check{X}$

CFT-version of MS suggests:

$\exists$ Top. branes in A-model dual to D0

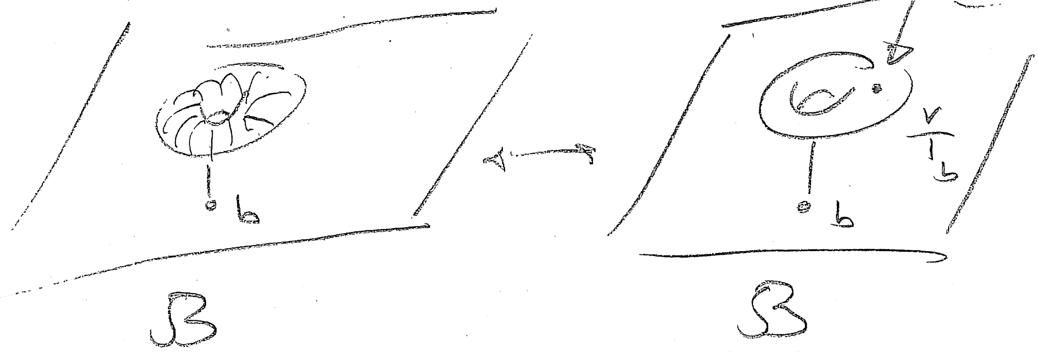
Branes in A-model: Objects in $Fuk(X)$
(L, ∇) ∇: Unit. flat. conn.

*) $\omega|_L = 0$, e.g. $\dim_{\mathbb{R}} L = 3$

$$\omega = \sum db_i \wedge d\theta^i,$$

e.g. $L_b = \{ x \in X \,;\, b_i = \text{const.} \}$

Conjecture (SYZ): $L_b \simeq \check{T}_b$



Mod. sp. of D0 on A-side:

- Choice of $b_i \in B$, $b_i \overset{(*)}{\sim} e^{-\int_{\beta_i} \omega}$
- Choice of $\nabla \sim Hol_{L_b}(\nabla) \sim (\theta_i)$

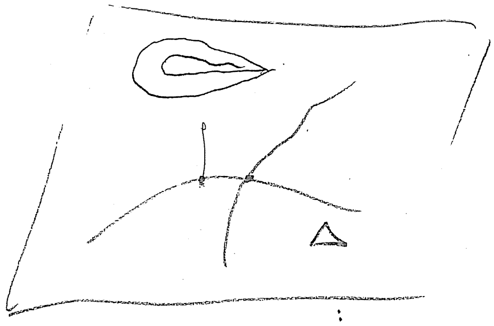
(*) $\beta_i \in H_2(X, L_b)$: $\int_{\beta_i} \omega$ - Area of disk with boundary on Lagrangian L_b

Mirror symm. predicts corresp. between

pairs (L, ∇) $\longleftrightarrow$ pts. on $\check{X}$

Spl. mfd. X cplx. mfd. $\check{X}$

Difficulty: Singular fibers, how to glue



$\int_{\beta_i} \omega = 0$

$\rightsquigarrow$ Divisor $\Delta \subset \mathcal{B}$

Ansatz: (Re-)construct cplx. struct by defining (local?) holom. coords. for $\check{X}$:

$$z_i = z_{\beta_i}(L, \nabla) = e^{-\int_{\beta_i}\omega} \, \mathrm{Hol}_{\partial\beta}(\nabla) \qquad (z)$$

Problem: z_i not single valued around Δ

Refine ansatz:

1) Break $\mathcal{B}_0 = \mathcal{B} \setminus \Delta$ into local charts
2) Use (z) to define coords in local charts
3) Find transition fct. between chart making $\mathcal{M}_{(L,\nabla)}$ into cplx. mfd. $\check{X}$

Very informal description of proposed answer to 1), 3): (Fukaya, Kontsevich-Soibelman, Auroux, Gross-Siebert)

Ad 1): Break $\mathcal{B}$ into chambers separated by walls:

Wall: Locus in $\mathcal{B}$ where holomorphic disks (with Maslov index $\mu = 0$) exist

Example:

$X = \mathbb{C}^2 \setminus D$, $D = \{(x,y) \in \mathbb{C}^2 ; xy = \varepsilon\}$

$\omega = \frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y})$

I: $\Omega = \dfrac{dx \wedge dy}{xy - \varepsilon}$ holom. (in x,y) vol. form

Lemma:

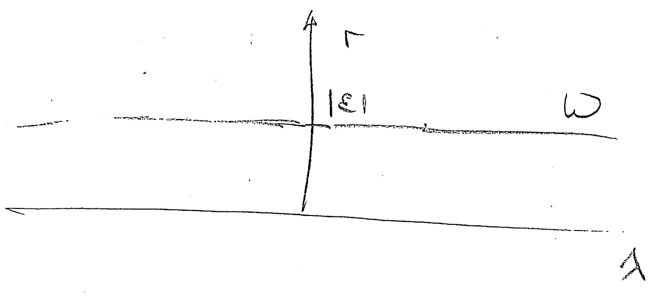
$T_{r,\lambda} = \{(x,y) \in \mathbb{C}^2, |xy - \varepsilon| = r, \mu_{S^1}(x,y) = \lambda\}$

$\frac{1}{2}(|x|^2 - |y|^2)$

are (special) Lagrangian Tori

Singular at $(r,\lambda) = (|\varepsilon|, 0)$.

Wall in $\mathcal{B}$ at $r = |\varepsilon|$



Proof uses projection f: $(x,y) \longrightarrow xy$

Assume there exists holom. map

$u : (D, \partial D) \to (X, T_{r,\lambda})$.

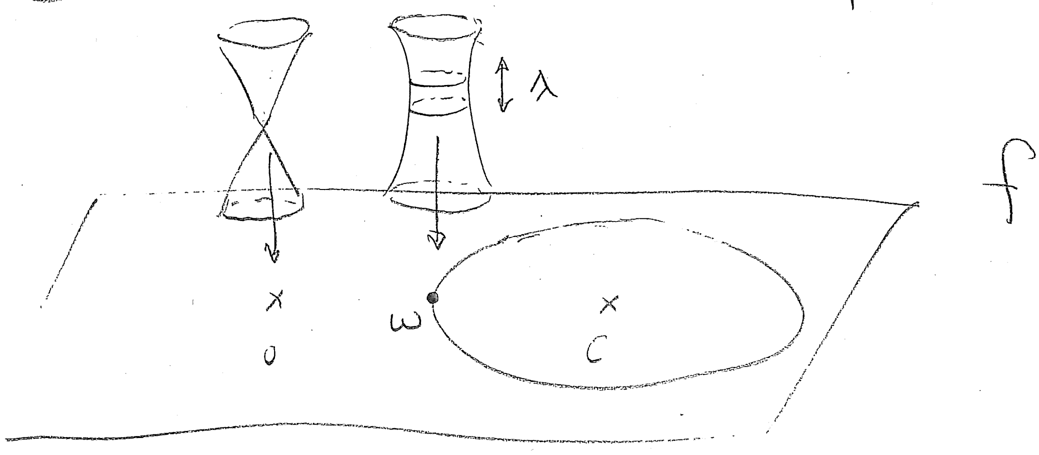
Consider $g := f \circ u - \varepsilon : D \mapsto \mathbb{C} \setminus \{0\}$ holom.

on ∂D : $|g| = r =$ const.

Maximum principle : Since g does not vanish on D, Max and Min of g on ∂D

$\Rightarrow$ $g =$ const.

So $D \sim$ fiber of f over pt. w



Fibres of f $\sim$ $S^1 \times \mathbb{R}$ for $r \neq |\varepsilon|$

" " " $\sim$ union of discs $r = |\varepsilon|$.

$\Rightarrow$ Wall ($\exists$ holo disk with μ-index 0) at $r = |\varepsilon|$.

Coords. in chamber I ($r > |\varepsilon|$) ?

Consider disks touching $xy = \varepsilon$ ($\rightsquigarrow \mu = 2$)

$\exists$ two types of such disks

($\sim D(r_1) \times \{y\}$, $\{x\} \times D(r_2)$), defining classes β_1, β_2 in $H_2(X, L_b)$

$\rightsquigarrow$ coords. $z_i = z_{\beta_i} = e^{-\int_{\beta_i} \omega} \, Hol_{\partial\beta_i}(\nabla)$

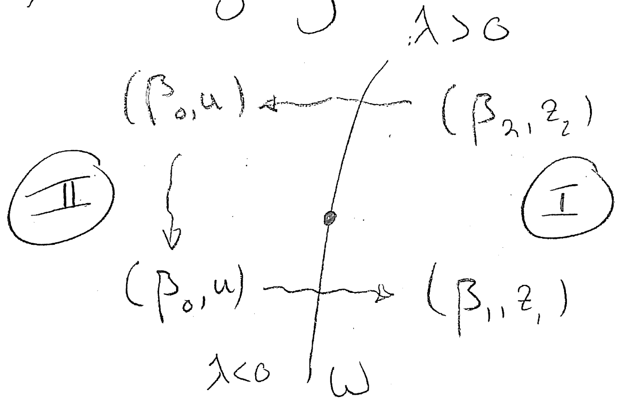
*) w: coordinate associated to $\mu = 0$ class α

Chamber II $(r < |\varepsilon|)$: Supplement by w *)

Only one such class β_0 $\rightsquigarrow$ coord. u. ✓

In order to define gluing, continue disks / coordinates across wall:

Naive gluing:



inconsistent !

(monodromy around Δ)

Way out: Define corrected gluing:

$\lambda > 0 \qquad u = z_2 (1 + w)$

$\lambda < 0 \qquad u = z_1 (1 + w^{-1})$

Noting that $w = z_1 / z_2$ $\rightsquigarrow$ consistency !

Tip of an iceberg:

- Corrected gluing cluster-like !
- More generally:

$$z_\gamma^{I} = z_\gamma^{II} \, f_{\gamma,\alpha}(z_\alpha) \qquad \left(\begin{array}{c} \text{count of} \\ (\mu = 0)\text{-disks} \end{array} \right)$$

($f_{\gamma,\alpha} = 1 + z_\alpha + \ldots$)

More conceptual approach: Theta-fcts.
(here: superpotential)

$$\vartheta_\iota(L,\nabla) := \sum_{\beta \in H_2(X,L)} N_\beta \, z_\beta(L,\nabla)$$

ι: where/how u allowed to touch singularities at D

N_β: (virtual) count of holom. discs $u: (D,\partial D) \to (X,L)$

<u>Claim</u>: ϑ_ι globally defined function on cluster variety defd. by corrected gluing.

$\rightsquigarrow$ Ring of alg fcts. on $\check{X}$,
$\check{X}$ as <u>alg.</u> <u>variety</u>!