Strominger-Yau-Zaslow mirror approach in a nutshell

V a real vector space $\left(\simeq \mathbb{R}^n_{y_1,\dots,y_n}\right)$ $\quad i=\sqrt{-1}$

TV tangent bundle has natural complex structure $\quad TV \simeq \mathbb{C}^n$

$$w_j = x_j + i\cdot y_j \qquad x_j := dy_j$$

T^*V cotangent bundle has natural symplectic structure

$$\omega := \sum_{j} dx_j^* \wedge dy_j \qquad x_j^* := \frac{\partial}{\partial y_j}$$

Add a flat framing $\Lambda \subseteq TV$ is an integer lattice in each tangent space

If $TV = \underset{y_j}{V} \times \underset{x_j}{V}$, e.g. $\Lambda = V\times\mathbb{Z}^n$

Define a quotient $X := TV/\Lambda$ is a complex manifold

In fact $X \simeq (\mathbb{C}^*)^n = V\times V/\Lambda \quad \mathbb{C}^* = \mathbb{C}\setminus\{0\}$

with coordinates $z_j = \exp\left(2\pi i\cdot(x_j + i y_j)\right)$ $\quad 0\to\mathbb{Z}\to\mathbb{C}\xrightarrow{\exp(2\pi i)}\mathbb{C}^*\to 0$ exact

$$= \exp(2\pi i\cdot w_j)$$

$X\to V$ projection is a real torus fibrations
fibres are V/Λ

$\check{X} = T^*V/\Lambda^*$ is a symplectic manifold

projection $\check{X}\to V$ is Lagrangian torus fibration

$\check{X} = \underset{y_j}{V}\times \underset{x_j^*}{V^*}/\Lambda^*$ $\qquad \omega|_{\text{fibre}} = 0$ because $y_j = \text{const} \Rightarrow dy_j = 0$

$\Leftrightarrow$ fibre

We see that T-duality relates a natural complex structure on X to a natural symplectic structure on the T-dual $\check{X}$.

How about the complementary parts?

X Kähler $\leftrightarrow$ $\check{X}$ Kähler (i.e. T-duality of Kähler mfds)

Add a potential to V: $K : V \to \mathbb{R}$ C^∞, convex

On X, get a Kähler form by $\omega_X := 2i \cdot \partial\bar{\partial} K$

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "K"] & \\ V \arrow[r] & \mathbb{R} \end{tikzcd}$$

Automatically $X \to V$ is Lagrangian.

On $\check{X}$, we obtain a complex structure (rendering $\check{X}$ Kähler) via coordinates

$$w_j^* = x_j^* + i \cdot \underbrace{\frac{\partial}{\partial y_j} K}_{=: y_j^*} \qquad x_j^* = \frac{\partial}{\partial y_j} \qquad y_j^* : \check{X} \to V \xrightarrow[\frac{\partial}{\partial y_j} K]{} \mathbb{R}$$

This gives T-duality for Kähler manifolds.

One can use K to obtain another atlas of affine coordinates on V

$$y_j^* = \frac{\partial}{\partial y_j} K$$ declare these to be affine coordinates (new affine structure)

With the y_j^*, the roles in the constructions swap complex & symplectic.

Remark

a) Get $\check{K}$ by Legendre transform: $\check{K}(y^*) = \max_y \{ \langle y, y^* \rangle - K(y) \}$

b) We have $y_j^* = y_j$ if $K = \frac{1}{2}(y_1^2 + \ldots + y_n^2)$ and then $K = \check{K}$.

Example $\mathbb{P}^2 = \underbrace{(\mathbb{C}^3 \setminus \{0\})}_{\{(z_1, z_2, z_3)\}/\text{scaling}} / \mathbb{C}^* \supseteq (\mathbb{C}^*)^3 / \mathbb{C}^* \simeq (\mathbb{C}^*)^2$

The projective plane has a well known Lagrangian torus fibration

$$\mathbb{P}^2 \xrightarrow{\pi} \mathbb{R}^2 \leftarrow \text{coordinates } y_1^*, y_2^*$$

$$(z_1,z_2) \longmapsto \left(\frac{|z_1|^2}{1+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{1+|z_1|^2+|z_2|^2}\right)$$

Surjects onto standard triangle: Convex hull $(0, e_1, e_2)$

$$(\mathbb{C}^*)^2 = \pi^{-1}(\text{interior})$$

This example arises as a compactification of the previous construction of $X \cong (\mathbb{C}^*)^2$

using $K = 1 + e^{2y_1} + e^{2y_2}$ $\quad K: \mathbb{R}^2 \to \mathbb{R}$

$e^{y_j} = |z_j| = |\exp(2\pi i(x_j + i y_j))|$ up to scaling y_j by -2π

Why do we project to an interior of triangle instead of all of $\mathbb{R}^2$?

Taking $y_j^* = \frac{\partial}{\partial y_j} K$ maps $\mathbb{R}^2$ diffeomorphically to interior of the standard triangle

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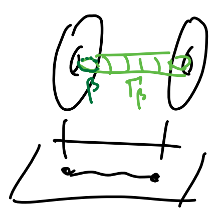
Towards scattering diagrams

Jörg already studied affine coordinates in singular torus fibrations.

$X \to B$ Lagrangian $\qquad \beta \in H_1(\text{fibre}, \mathbb{Z}) = \Lambda$

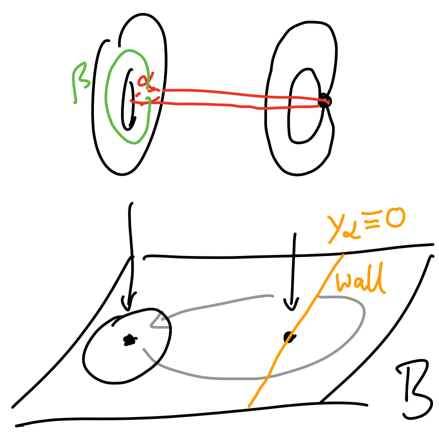
$$y_j = -\int_{\Gamma_\beta} \omega$$

$$z_\beta = \exp(-y_j) \cdot \underbrace{\mathrm{hol}_{\partial\beta}(\nabla)}_{\exp(i x_j)}$$



Γ_β = chain traced by β as fibre moves along a path in the base

Dehn twist situation



$Y_\alpha = -\int_{\Gamma_\alpha} \omega$ ← this is global coordinate

$Y_\beta = -\int_{\Gamma_\beta} \omega$ ← not globally well defined

$\alpha \mapsto \alpha$
$\beta \mapsto \beta + \alpha$

monodromy $Y_\alpha \mapsto Y_\alpha$
$Y_\beta \mapsto Y_\beta + Y_\alpha$

$z_\alpha = \exp(2\pi i (x_\alpha + i y_\alpha))$

$z_\beta = \exp(2\pi i (x_\beta + i y_\beta))$

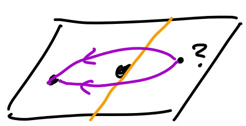
$z_{\alpha+\beta} = \exp(2\pi i (x_{\alpha+\beta} + i y_{\alpha+\beta}))$

$z_{\alpha+\beta} = z_\alpha \cdot z_\beta$

Solution to non-global coordinate

use $z_\beta + z_{\beta+\alpha} = \underline{z_\beta(1+z_\alpha)}$

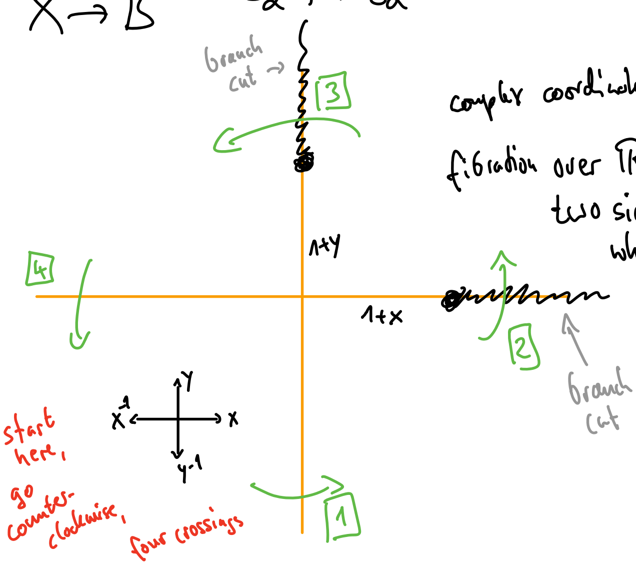
as a "continuation" of z_β beyond the wall.



Sum of exp of chain integrals

$z_\beta \longmapsto \underline{z_\beta(1+z_\alpha)}$
$z_\alpha \mapsto z_\alpha$

$X \to B$



complex coordinates, fibration over $\mathbb{R}^2$

two singular fibres whose walls are coordinate axes

Lets study how the first complex coordinate X transforms when crossing several walls:

walk counterclockwise around the origin

Let us transform x: (y goes similar)

$$x \overset{\boxed{1}}{\longmapsto} (1+y)^{-1}x \overset{\boxed{2}}{\longmapsto} \left(1+(1+x)^{-1}y\right)^{-1}x$$

$$\overset{\boxed{3}}{\longmapsto} \left(1+(1+(1+y)x)^{-1}y\right)^{-1}\cdot(1+y)\cdot x$$

$$\overset{\boxed{4}}{\longmapsto} \left(1+(1+(1+(1+x)y)x)^{-1}(1+x)y\right)^{-1}\cdot(1+(1+x)y)x$$

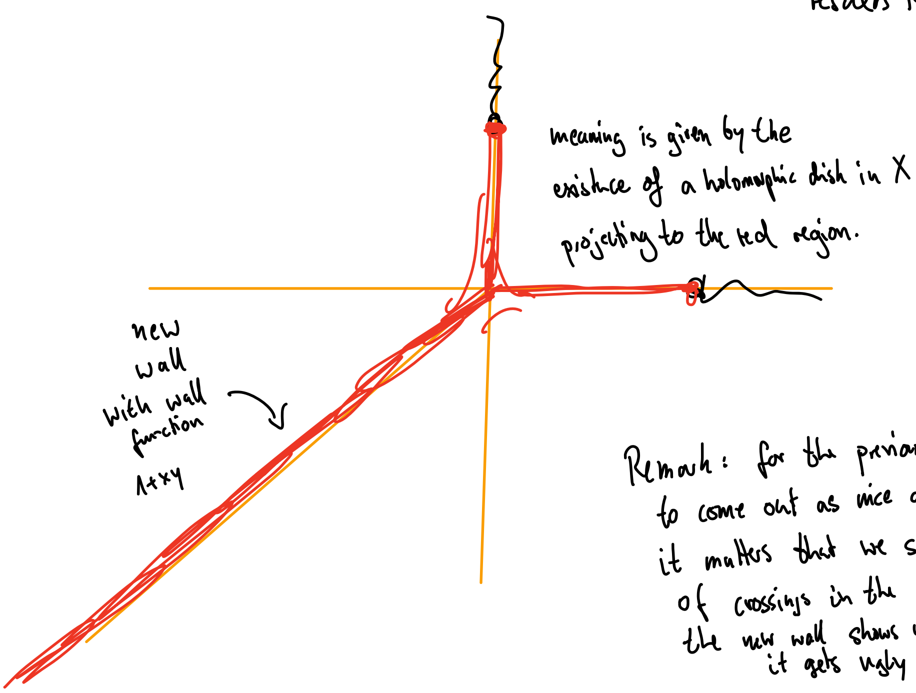
$$= \left(1+((1+x)+(1+x)yx)^{-1}(1+x)y\right)^{-1}\cdot(1+y+xy)x$$

$$= \left(1+(1+xy)^{-1}y\right)^{-1}\cdot(1+y+xy)x$$

$$= (1+xy+y)^{-1}\cdot(1+xy)\cdot(1+y+xy)\,x$$

$$= (1+xy)\cdot x$$

Upshot: if you insert an extra wall with wall function $1+xy$ in the bottom left quadrant where we started then (after crossing that also) the loop of coordinate transformations results in identity.



Remark: for the previous computation to come out as nice as it did, it matters that we start the loop of crossings in the quadrant where the new wall shows up. Otherwise it gets ugly.

Kontsevich-Soibelman-Lemma:

Given any set of initial walls, there is a (essentially unique) set of extra walls to insert so that every path ordered composition yields the identity (as in computation above)

Theorem (Lin et al) ← consequence of hyperkähler rotation

There is a special Lagrangian fibration of $\mathbb{P}^2 \setminus E$

↖ smooth ell. curve

The corresponding affine structure has three singular fibres (i.e. discriminant points) and was first given in

Carl-Pumperla-Siebert:

https://arxiv.org/abs/2205.07753

Its Scattering diagram was computed in

https://arxiv.org/pdf/2204.12249.pdf#page49

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