Cluster Coordinates for Moduli Spaces of Flat Connections:

$SL(2,\mathbb{C})$ on Riemann surfaces

PLAN:

1. Introduction & motivation
2. The moduli space of flat connections
3. Triangulations and their associated cluster coordinates
4. Triangulation flips and cluster mutations:
5. Application to the BPS spectrum of 4d $\mathcal{N}=2$ QFT

Main References:

- **[GMN11]** : Gaiotto, Moore, Neitzke, 2011,
  "Wall-Crossing, Hitchin Systems, & the WKB Approximation", [0907.3987] (1), (3), (4), (5)
  §1, §2, **§5**, §6, §7
- **[FST]** : Fomin, Shapiro, Thurston, 2007,
  "Cluster Algebras and Triangulations Part I : Cluster Complexes", [math/0608367] (3), (4)
  §2, **§4**
- **[CLT]** : Coman-Lohi, Longhi, Teschner, 2020,
  "From quantum curves to topological string partition functions II", [2004.04585] (4)
  §4.3.1, **§5.6.2, §5.6.3**
- **[C]** : Coman-Lohi, Ph.D. thesis,
  "On generalisations of the AGT correspondence for non-Lagrangian theories of class S", PDF (2)
  §2, §4, references therein

Complementary References:

- **[FG03]** : Fock, Goncharov, 2003,
  "Moduli spaces of local systems and higher Teichmüller theory", [math/0311149]
- **[Ip17]** : Ip 2017,
  "Introduction to Cluster Algebras", Lecture Notes, Part V
- **[W12]** : Williams, 2012,
  "Cluster algebras: an introduction", [1212.6263]
- **[IN14]** : Iwaki, Nakanishi,
  "Exact WKB analysis and cluster algebras", [1401.7094]
- **[A18]** : Allegretti, 2018,
  "Voros symbols as cluster coordinates", [1802.05479]

(1) Intro & motivation:

DEFINE:
- a Riemann surface C' of genus g, with punctures $\{P_i\}_{i=1}^{n}$
- a flat $SL_2(\mathbb{C})$ connection A on C' with "regular" singularities at punctures:

$$[d+A, d+A] = 0, \qquad A = \mathcal{O}\left(\frac{1}{z-z_i}\right)dz \quad \text{near } z_i \equiv z(P_i)$$

- the moduli space of flat connections

$$\mathcal{M} := \{A \text{ flat}\} / \{\text{gauge transformations}\}$$

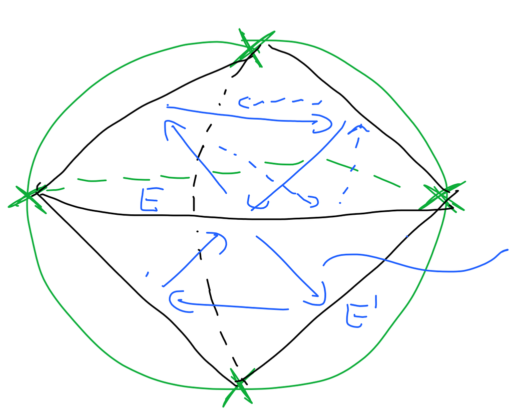
$$\cong \mathrm{Hom}\left(\pi_1(C), SL_2(\mathbb{C})\right) / SL_2(\mathbb{C})$$

$\cup$ holonomies $Pe^{\oint_\gamma A}$

fix conj. class of monodromies @ punctures $\longrightarrow$ $\mathcal{M}$ = symplectic manifold

GOAL: Associate to any TRIANGULATION of C a CLUSTER SEED and CLUSTER COORDINATES on $\mathcal{M}$

T



$C_{0,4}$

$\delta(T)_0 = \{E\}$

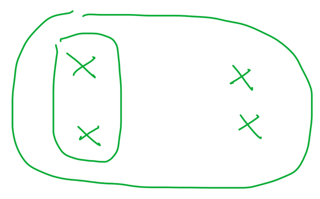
$b_{EE'}$, $\{X_E^T\}_{E \in S(T)_0}$

MOTIVATIONS:

M1: Geometric realization of cluster algebras:

$$T \to T' : \quad X_E^T \to X_{E'}^{T'} = \mu(X_E^T) \quad , \quad \{X_E^T, X_{E'}^T\} = b_{EE'} X_E^T X_{E'}^T$$

Ex: Infinite mutation sequences associated to holonomies of non-contractible cycles in C.



M2: $\mathcal{N}=2$ SUSY gauge theory on $\mathbb{R}^3 \times S^1_R$ of "class S"

$$\mathcal{A} = A + R(\zeta^{-1}\varphi + \zeta\bar{\varphi})$$

in $\mathcal{N}=2$ vector multiplet.

$SU(2)$ connection

complex structure of $\mathcal{M}$ = hyperkähler

breaks $SU(2)$ to $U(1)$

→ Coulomb branch vacua labeled by $u\,dz^2 := \frac{1}{2}\operatorname{tr}\varphi^2$, $u \in \mathcal{B}$

→ Hilbert space of BPS states at $u \in \mathcal{B}$:

$\frac{1}{2}\operatorname{tr}\varphi^2 = \lambda_{SW}^2$

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma_u} \mathcal{H}_{\gamma,u} \quad , \quad \Gamma_u = \Gamma_u^e \oplus \Gamma_u^m \cong H_1(\Sigma_u^{SW}, \mathbb{Z})_{odd}$$

odd under $\lambda \to -\lambda$.

GOAL: determine $\dim \mathcal{H}_{\gamma,u}$ (or the index $\Omega_{\gamma,u}$)

METHOD: associate cluster coordinates to $u, \gamma, \vartheta \in [0, 2\pi)$:

$$(u,\vartheta) \leftrightarrow T(u,\vartheta)$$
$$\gamma \leftrightarrow E_\gamma \in T(u,\vartheta)$$

↳ Evolve $T(u,\vartheta) \to T(u,\vartheta+\pi)$ and compute

$$T(u,\vartheta+\pi) = \prod_{\gamma:\ \arg(-Z_{\gamma,u}) \in [\vartheta, \vartheta+\pi]} K_\gamma^{\Omega_{\gamma,u}} \cdot T(u,\vartheta)$$

sequences of mutations

index counting BPS states

$Z_{\gamma,u} = \oint_\gamma \sqrt{\frac{1}{2}\operatorname{tr}\varphi^2}$: charges of unbroken $U(1)$

Wall-crossing: $u \longrightarrow u'$

$$\prod_\gamma K_\gamma^{\Omega_{\gamma,u}} \longrightarrow \prod_\gamma K_\gamma^{\Omega_{\gamma,u'}} \neq \Omega_{\gamma,u}$$

UPSHOT: Mutations of $T(u,\vartheta)$ control the BPS spectrum.

Note: Connection with Ivan's upcoming talk:

- a general $\mathcal{N}=2$ SUSY QFT on $\mathbb{R}^3 \times S^1_R$ has a low energy description as a $\mathbb{R}^3 \to \mathcal{M}$ σ-model, where $\mathcal{M}$ hyperkähler with complex structure $\zeta \in \mathbb{CP}^1$

- [GMN09] construct coordinates $X_\gamma^{RH}(\zeta)$ on $\mathcal{M}$ defined by:

  P1: X_γ^{RH} are Darboux ← see [GMN'11, eq. (1.5)]

  P2: $X_\gamma^{RH} \overset{R\to\infty}{\sim} X_\gamma^{sf}$ AND $X_\gamma^{RH} \overset{\zeta\to 0}{\sim} X_\gamma^{sf}$

  P3: $X_\gamma^{RH}(e^{i\pi}\zeta) = \prod_{\gamma_0:\ -\arg Z_{\gamma_0}(u) \in [\arg\zeta, \arg\zeta+\pi]} K_{\gamma_0}^{\Omega_{\gamma_0,u}} \cdot X_\gamma^{RH}(\zeta)$, and

  $X_\gamma^{RH}(\zeta)$ is otherwise piecewise holomorphic.

- $X_\gamma^{RH}(\zeta)$ are constructed as solutions to a Riemann-Hilbert problem ⟺ TBA equation.

↳ RESULT: If the theory is "class S" and R is sufficiently large, then

$$X_\gamma^{RH} = X_{E_\gamma}^{T(u,\vartheta)}\Big|_{\vartheta=\arg\zeta}$$

and K_{γ_0} in $\boxed{P3}$ is a sequence of mutations associated to $E_{\dots}$

(2) The moduli space of flat connections:

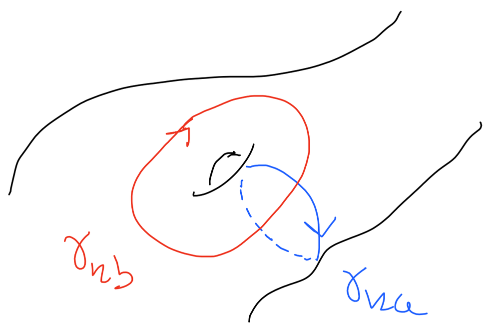
2.1 : Definition (generators & relations)

- flatness : $[d+A, d+A] = dA + A \wedge A = 0$
- reg. singularities @ P_i : $A = \left(\frac{T_i}{z-z_i} + \mathcal{O}(1) \right) dz$

↳ for $\circlearrowleft P_i$ γ_i : $\boxed{M_i = P e^{\oint_{\gamma_i} A} = e^{2\pi i T_i}}$ monodromy matrices @ P_i

Require: $M_i = g \, \mathrm{diag}(m_i, m_i^{-1}) \, g^{-1}$ for some $g \in SL_2(\mathbb{C})$, m_i = fixed

Define: for every $k = 1, \dots, g$



$A_k := P e^{\oint_{\gamma_{ka}} A}$ $\qquad$ $B_k := P e^{\oint_{\gamma_{kb}} A}$.

$\Rightarrow$ action of gauge trnsftms reduces to adjoint action of $SL_2(\mathbb{C})$ on M_i, A_k, B_k.

$$\Rightarrow \boxed{\mathcal{M} = \left< M_i, A_k, B_k \,\middle|\, \prod_{i=1}^{n} M_i = \prod_{k=1}^{g} A_k B_k A_k^{-1} B_k^{-1} \right> \Big/ SL_2(\mathbb{C})}$$

Dimension : $\dim_{\mathbb{C}} \mathcal{M} = (\underset{M_i}{n} + \underset{A_k, B_k}{2g}) \dim SL_2 - n - 3 - 3$ $\quad [M_i] = \left[\begin{pmatrix} m_i & 0 \\ 0 & m_i^{-1} \end{pmatrix} \right]$, relations, $SL(2, \mathbb{C})$ conjugation

$$\boxed{\dim_{\mathbb{C}} \mathcal{M} = 2n + 6g - 6.}$$

2.2 : Symplectic structure :

Atiyah-Bott: $\Omega = \int_{C} \frac{1}{2} \mathrm{tr} \, \delta A \wedge \delta A = \int_{C} dz d\bar{z} \; \delta A^{a}_{z\,b} \, \delta A^{b}_{\bar{z}\,a}$

$\delta A_z = \mathcal{O}(1)$ near $z = z_i$

Fact : Ω is non-degenerate. on $\mathcal{M}$

Goldman 1986 : If $L(\gamma) := \mathrm{tr} P e^{\oint_\gamma A}$, then $\Omega^{-1} = \{-,-\}$, with

$$\boxed{\{L(\gamma), L(\gamma')\} = \sum_{P \in \gamma \cap \gamma'} \varepsilon(P; \gamma, \gamma') \left\{ L(\gamma_P \circ \gamma'_P) - \tfrac{1}{2} L(\gamma) L(\gamma') \right\}.}$$

CLAIM: $\{X_E^T, X_{E'}^T\} = b_{EE'} X_E^T X_{E'}^T$.

Proof: either : compute δX_E^T in terms of δA [GMN 11, appendix B]

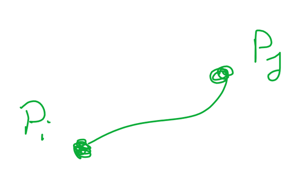
or : compute $L(\gamma)$ as a fctn of X_E^T [GMN 11, appendix A]

(3) Triangulations & their associated cluster coordinates:

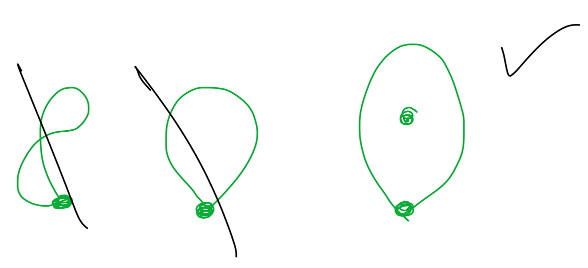
Note: I will consider C without boundary, but the discussion easily generalizes with {boundary edges} $\Longleftrightarrow$ {frozen cluster variables}.

3.1 Triangulations of Riemann surfaces:

Arc: Curve γ in C such that



- endpoints of γ are punctures
- γ may intersect itself ONLY at its endpoint
- γ is not contractible into punctures :



Compatible arcs: $\gamma \cap \gamma' \subset \{P_i\}_{i=1}^{n}$

Triangulation: $T = \{\gamma_E\}_{E=1}^{\#E}$ MAXIMAL collection of pairwise compatible arcs

Note: $\boxed{\#E = 3n + 6g - 6}$ independent of the choice of T

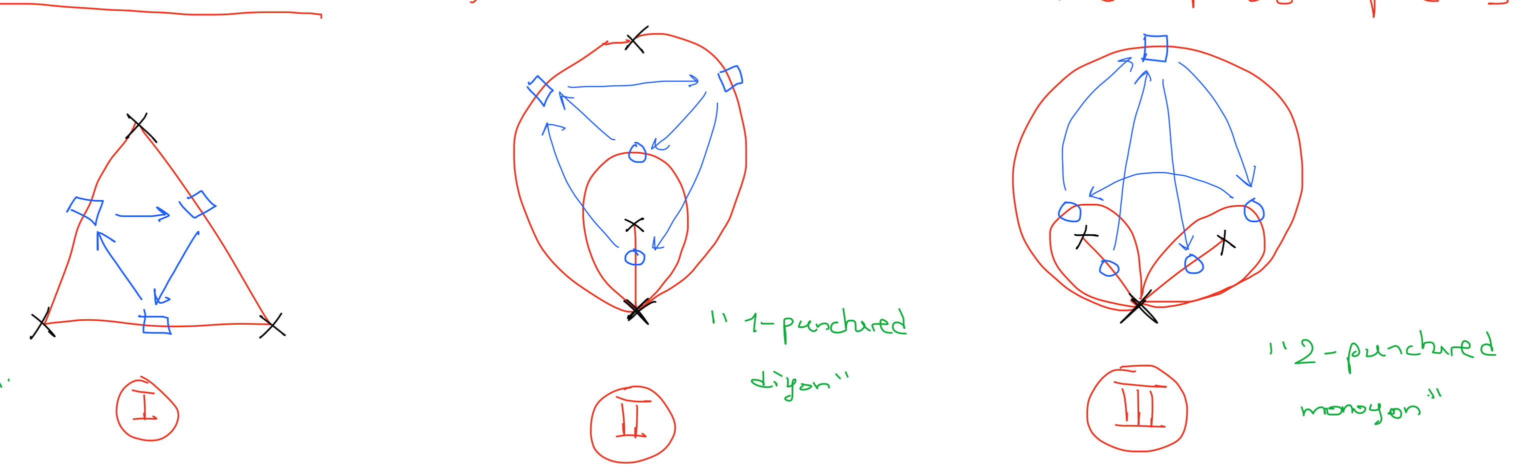
↳ $2\#E = 3\#F$, $\boxed{\chi = n - \#E + \#F = 2 - 2g}$

3.2 Cluster seed of a triangulation:

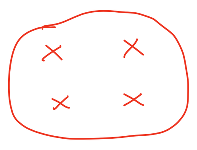
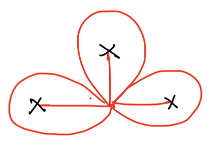
- Associate to any $T = \{\gamma_E\}_E$ a <u>quiver</u> $S = \{S_0, S_1\}$ such that

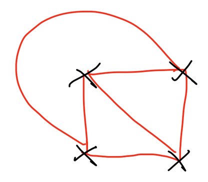
$$S_0 = \{E\}, \quad S_1 = \{E \xrightarrow{\times 0,1,2} E'\} \qquad \boxed{b_{EE'} := \#(E \to E') - \#(E' \to E)}$$

<u>Explicit construction</u> [FST, Remark 4.2]: Define three "puzzle pieces"



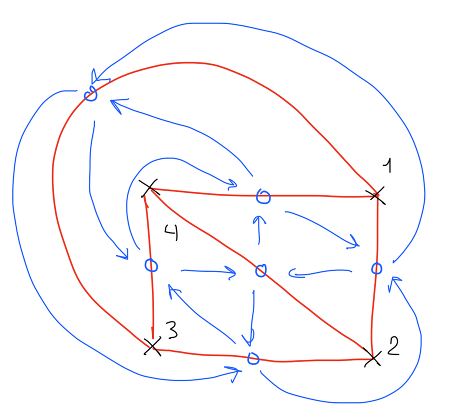
Then any cluster seed $S(T)$ is obtained by <u>gluing together</u> pairs of (one or more) OUTER edges "□" of copies of I, II, III, with ONE exception:

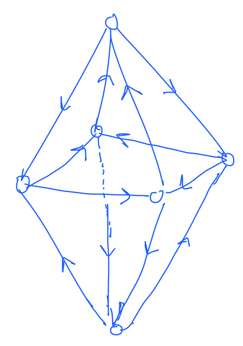
$C_{0,4} =$  , $T =$  , where

$S(T) = S\Big($  $\Big)$ with OPPOSITE orientation of arrows,

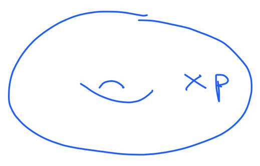
"tetrahedron triangulation"

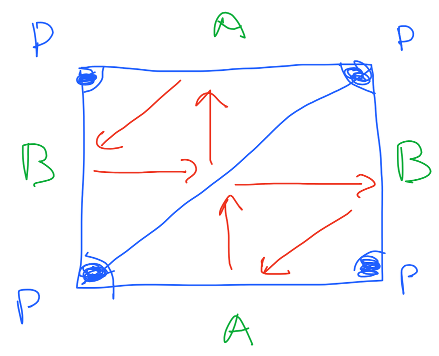
<u>Note</u>:

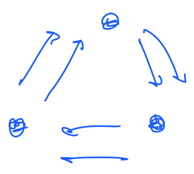


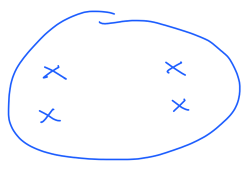
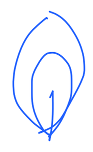
$=$


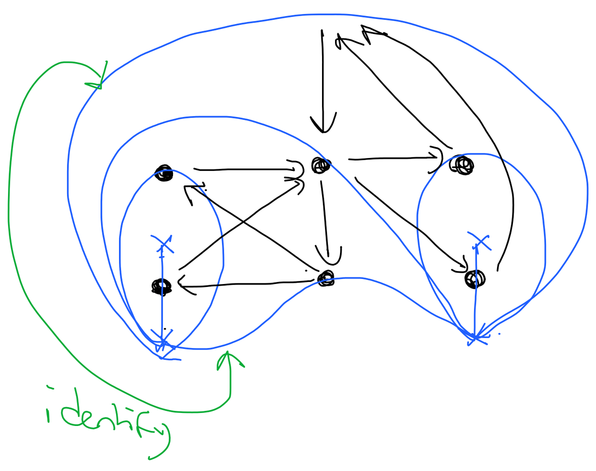
orientation of the 1-skeleton of the octahedron.

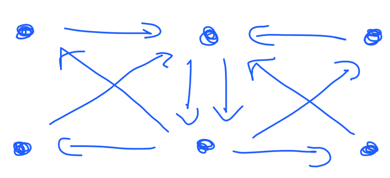
<u>Examples</u>: 1) $C = C_{1,1} =$  represent torus as $\mathbb{C}/\mathbb{Z}^2$:



$\xrightarrow[\text{edges}]{\text{identify}}$ $S =$  "Markov quiver"

2) $C = C_{0,4} =$  : Glue two copies of  along two edges:



$\longrightarrow$ $S =$  "type $D_4^{(1,1)}$" (Saito 1985)

<u>3.3</u> Fock - Goncharov coordinates:

<u>Cluster A- and X-variables</u>: Given a cluster seed S, we associate:

- $\{A_E\}_{E \in S_0}$, satisfying exchange relations

$$(\mu_A) \quad \boxed{\begin{aligned} A_E\, \mu_E(A_E) &= \prod_{E' : b_{EE'} > 0} A_{E'}^{b_{EE'}} + \prod_{E' : b_{EE'} < 0} A_{E'}^{-b_{EE'}} \\ \mu_E(A_{E'}) &= A_{E'} \end{aligned}}$$

- $\{X_E\}_{E \in S_0}$, satisfying exchange relations

$$(\mu_X) \quad \boxed{\begin{aligned} \mu_E(X_E) &= X_E^{-1} \\ \mu_E(X_{E'}) &= X_{E'}\left(1 + X_E^{\operatorname{sgn}(b_{EE'})}\right)^{b_{EE'}} \end{aligned}}$$

<u>Result</u>: If $\{A_E\}$ satisfy (μ_A), then $\left\{X_E := \prod_{E'} A_{E'}^{b_{EE'}}\right\}$ satisfy (μ_X).

Realization on $\mathcal{M}$:

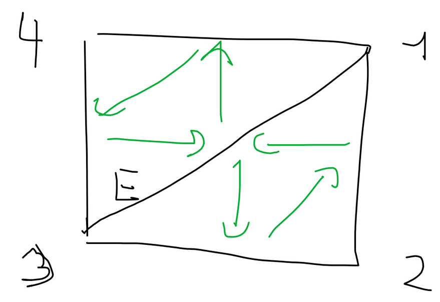
- Define <u>flat sections</u> $s_i : \mathbb{C} \to \mathbb{C}^2$ by:

$$(d+\mathcal{A})s_i = 0 \ , \quad M_i s_i = m_i s_i \ , \quad M_i = P e^{\oint_{\gamma_i} \mathcal{A}} \qquad (\mathcal{S})$$

→ $(\mathcal{S})$ uniquely defines the ray $\{\lambda s_i\}_{\lambda \in \mathbb{C}^\times}$ on a <u>simply connected</u> domain $U \subset \mathbb{C}$, $p_i \in \overline{U}$.

→ otherwise, s_i will have a <u>branch cut</u> around p_i.

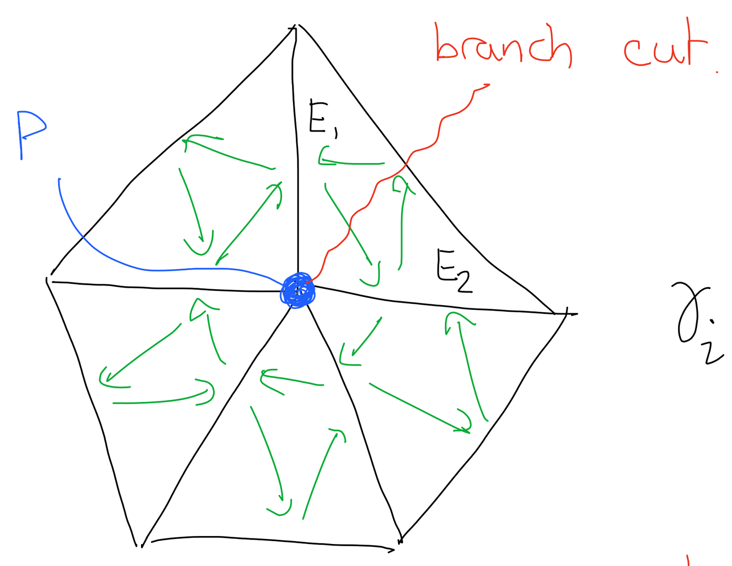
- In a quadrilateral:

$Q_E :=$ 

$$\boxed{\begin{aligned} A_E^T &:= \epsilon_{ab}\, s_1^a s_3^b = s_1 \wedge s_3 \\ X_E^T &:= \frac{(s_1\wedge s_2)(s_3 \wedge s_4)}{(s_2\wedge s_3)(s_4\wedge s_1)} \end{aligned}}$$

Monodromy constraint:

→ consider a triangulation where p_i is the endpoint of $E_1, \ldots, E_N$:



$$\gamma_i \subset \bigcup_{s=1}^{N} Q_{E_s}. \qquad X_{E_1}^T \to m_i X_{E_1}^T \qquad X_{E_2}^T \to m_i X_{E_2}^T$$

$$\hookrightarrow \boxed{\prod_{s=1}^{N} X_{E_s}^T = m_i^2}$$

<u>Check</u>: For any ideal triangulation:

$$\#\{E\} = 3n + 2g - 6 = \dim_{\mathbb{C}} \mathcal{M} + n$$

monodromy constraints

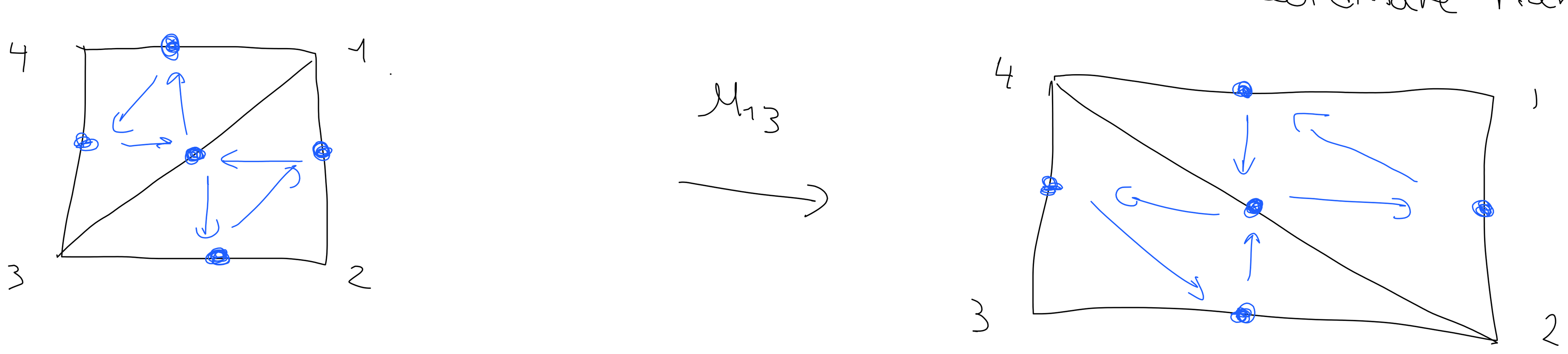
④ Triangulation flips & cluster mutations:

<u>4.1</u> General properties

- Exchange relations are based on $Gr(2,n)$ Plücker relations:

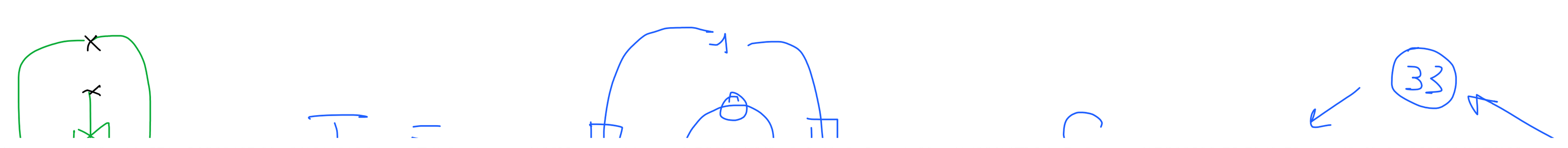
$$\boxed{(s_1\wedge s_3)(s_2 \wedge s_4) = (s_1\wedge s_2)(s_3\wedge s_4) + (s_2 \wedge s_3)(s_4 \wedge s_1)}$$

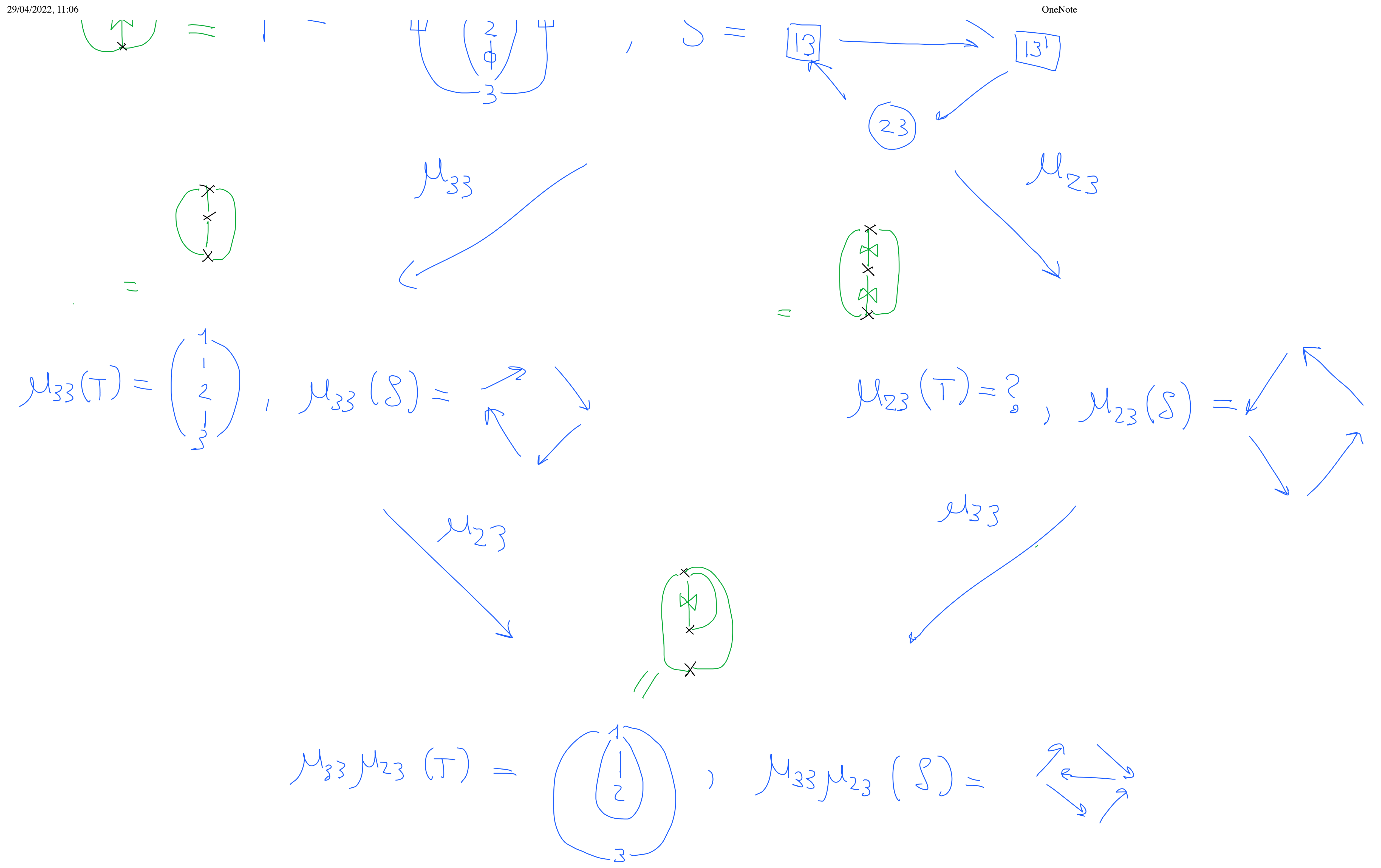
- Flip of triangulation ⟺ cluster mutation ⟺ coordinate transformation



<u>Limit case</u>: <u>Self-folded triangles & tagged flips</u>:

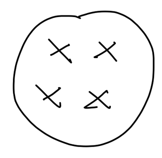
Consider puzzle piece (II), freeze the two outer nodes, and consider its exchange graph:

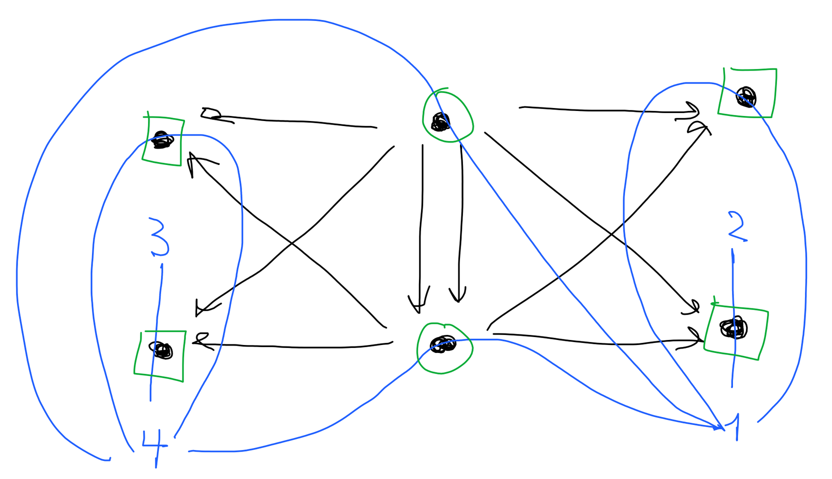


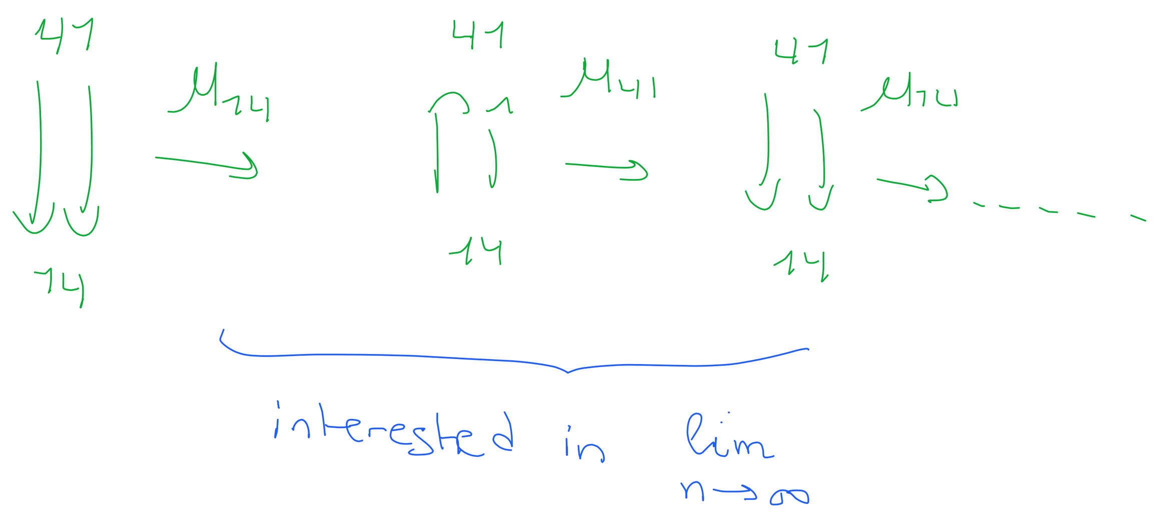


↳ resolution as flips of TAGGED triangulations [FST], [Ip 17]

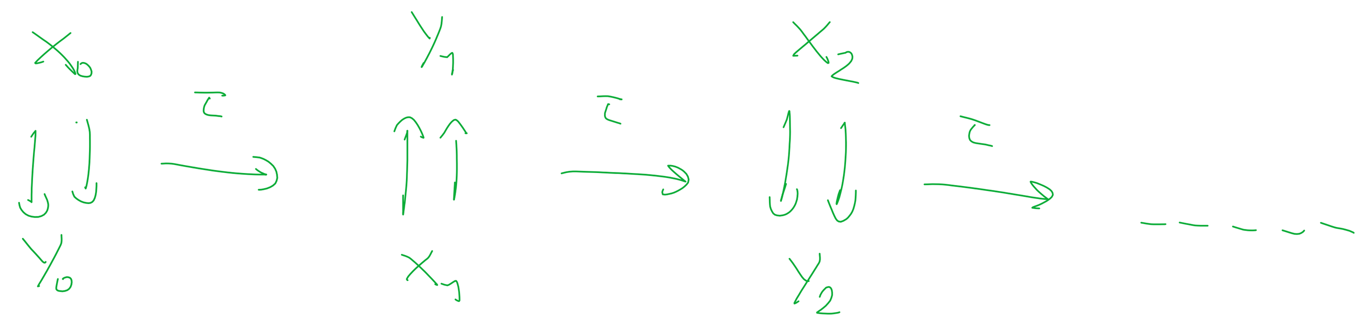
4.2 : Infinite mutation sequences & non-contractible (but non-trivial) cycles

- Consider the $D_4^{(1,1)}$ quiver for  :





Define:

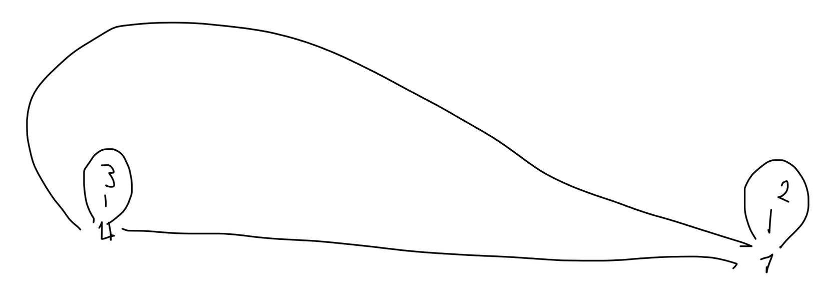


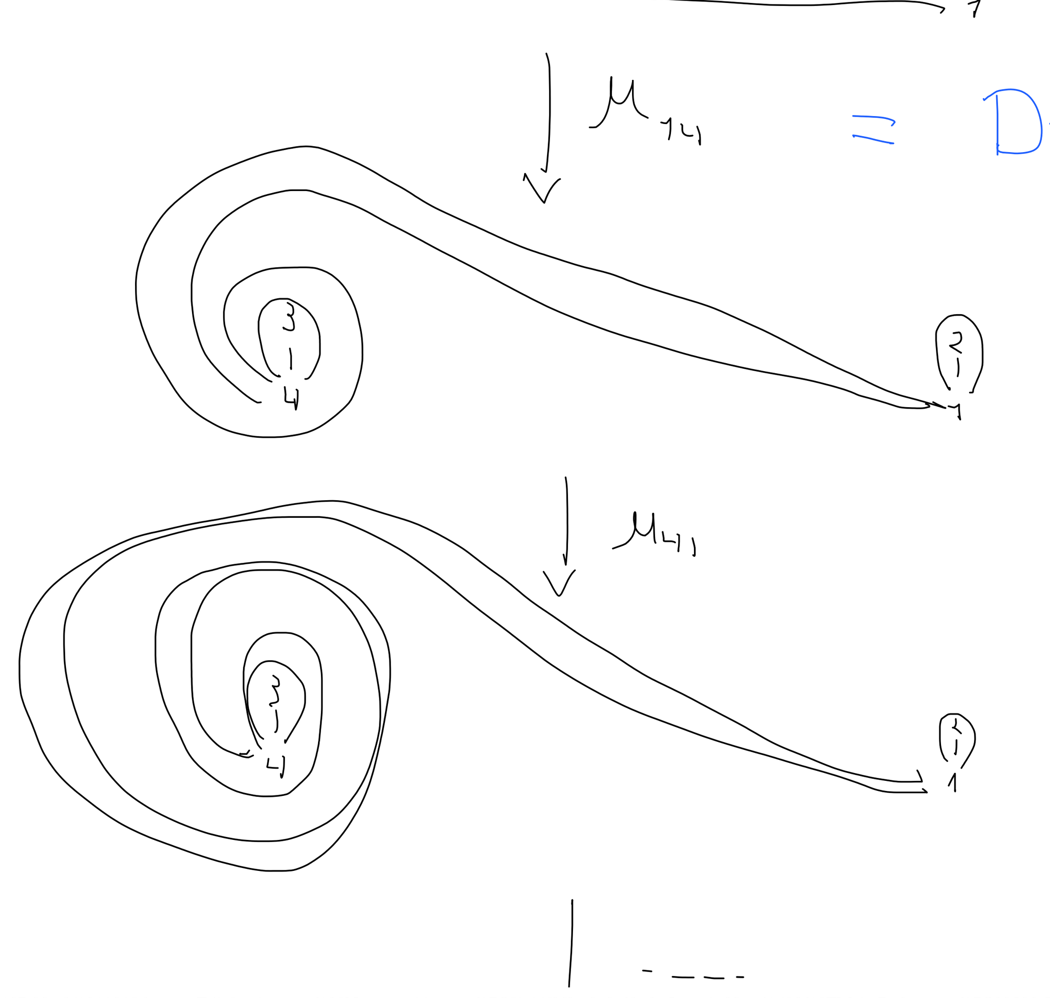
$\tau^n = \mu_{14}\mu_{41}\mu_{14}\dots$

$\Longrightarrow \quad X_{n+1} = Y_n^{-1}, \quad Y_{n+1} = X_n(1+Y_n^{-1})^{-2}$

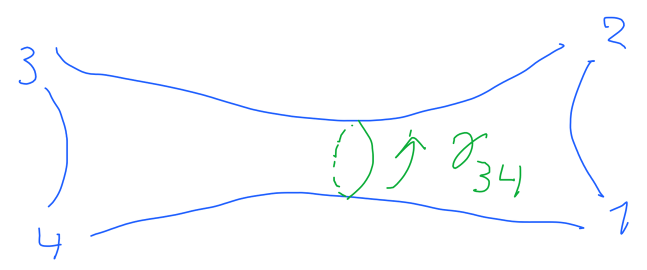
$\Longleftrightarrow \quad \boxed{X_{n+2}X_n = (1+X_{n+1})^2}$ cf for normalized A-variables $\boxed{A_{n+2}A_n = 1 + A_{n+1}^2}$

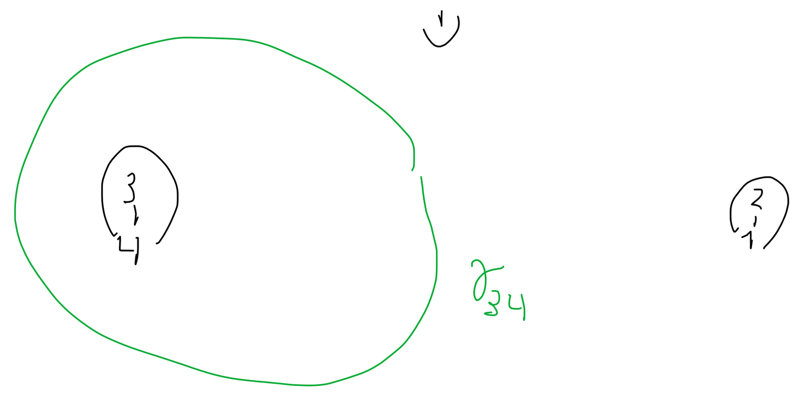
- Geometric interpretation:

$T =$ 



μ_{14} = DEHN TWIST around



$$\lim_{n\to\infty} \tau^n(T) =$$ 

↳ in FG coordinates: $X_n = \frac{C_{-n}}{K}$, $Y_n = \frac{K}{C_{1-n}}$,

$$C_n := (S_1 \wedge M^k S_4)^2, \quad M := \text{monodromy around } \gamma_{34}$$

$$K := A_{E_{34}} A_{E_{44}} A_{E_{12}} A_{E_{11}}$$

- Limit variables as Fenchel-Nielsen coordinates:

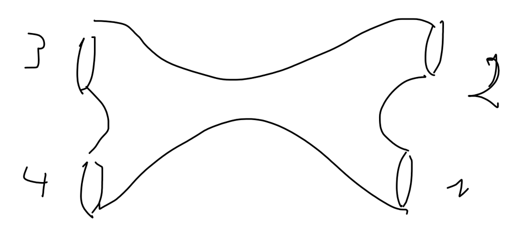
If $M = g \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix} g^{-1}$ and $|U| > 1$, it's easy to show that:

$$\boxed{\lim_{n\to\infty} X_n Y_n = \lim_{n\to\infty} \frac{C_{-n}}{C_{1-n}} = U^2} =: X_A^{T_\infty}$$

Similarly:

$$\lim_{n\to\infty} \frac{X_n}{(X_n Y_n)^n} = \frac{(S_1 \wedge P_- S_4)^2}{K} =: X_B^{T_\infty}$$

P_-: projector on to U^{-1}-eigenspace of M

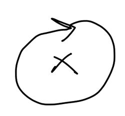
$X_A^{T_\infty}$, $X_B^{T_\infty}$ = Darboux coordinates associated to the  decomposition

↳ "Fenchel-Nielsen type" coordinates on $\mathcal{M}$ (see [CLT20])

Juggle: If we define $\tau^n = \mu_{14}\mu_{41}\dots$, $\tau^{-n} = \mu_{41}\mu_{14}\dots$

and $X_A^{T_{\pm\infty}} := \lim_{m\to\pm\infty} X_m Y_m$, $X_B^{T_{\pm\infty}} = \lim_{m\to\pm\infty} \frac{X_m}{(X_m Y_m)^m}$, then

$$\boxed{X_A^{T_{-\infty}} = \left(X_A^{T_{+\infty}}\right)^{-1} = U^{-2}, \quad X_B^{T_{-\infty}} = \left(X_B^{T_{+\infty}}\right)^{-1} (U - U^{-1})^{-4}}$$

Expectation: All CAs realized by surfaces with non-contractible cycles (outside of ) are mutation-infinite.

5 Application to the BPS spectrum of $d=4$, $\mathcal{N}=2$, SUSY gauge theories:

Recall: $\mathcal{A} = A + R(\zeta^{-1}\varphi + \zeta\bar{\varphi})$

Define: $\frac{1}{2}\operatorname{tr}\varphi^2 = u\,dz^2 := \lambda^2$

quadratic differential; u: coordinate on the Coulomb Branch $\mathcal{B}$ (moduli space of vacua); λ: Seiberg-Witten form

5.1: WKB triangulation:

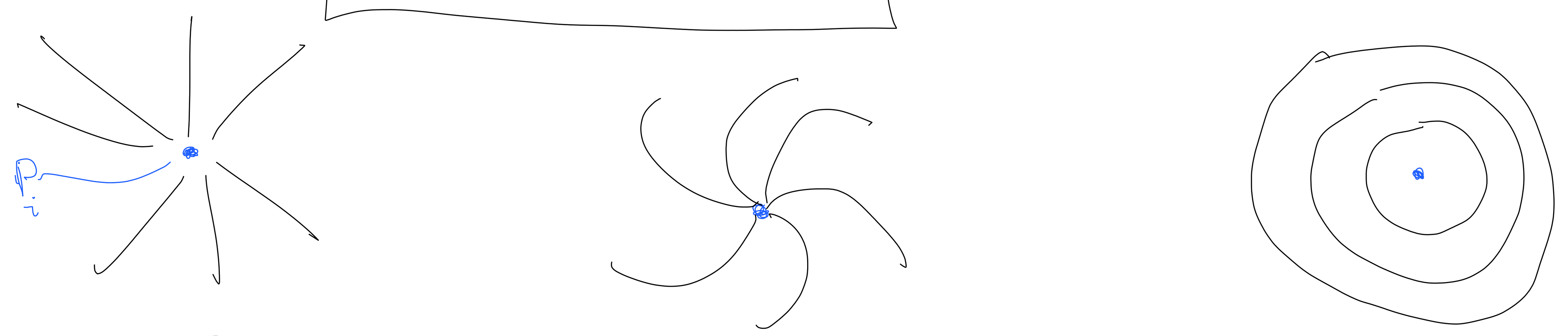
Goal: Associate canonical triangulation of C to any SW differential λ, @ any angle ϑ.

WKB curve: $\lambda[\partial_t] = \lambda_z \frac{dz}{dt} = \alpha e^{i\vartheta}$, $\alpha \geq 0$.

Behavior near critical points:

→ at a puncture $z(P_i) = z_i$: $\lambda = \left(\frac{m}{z - z_i} + \mathcal{O}(1) \right) dz$

↳ $\boxed{z(t) - z_i = z_0\, e^{\frac{\alpha}{m} e^{i\vartheta} t}}$



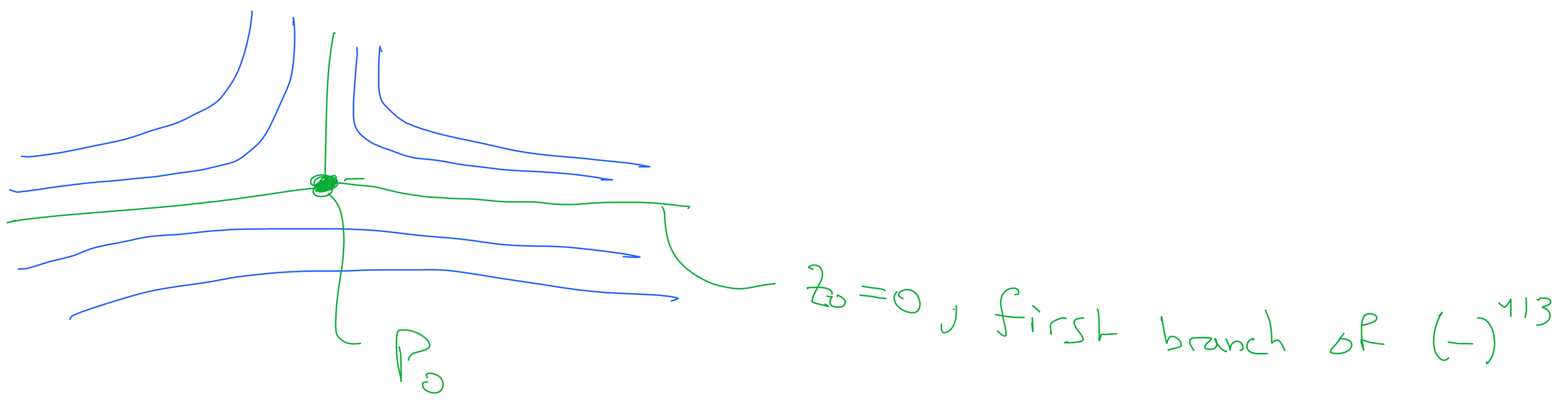
↳ $\frac{e^{i\vartheta}}{m} \in \mathbb{R}$ generic case $\frac{e^{i\vartheta}}{m} \in i\mathbb{R}$

⟹ punctures = ENDPOINTS of WKB curves.

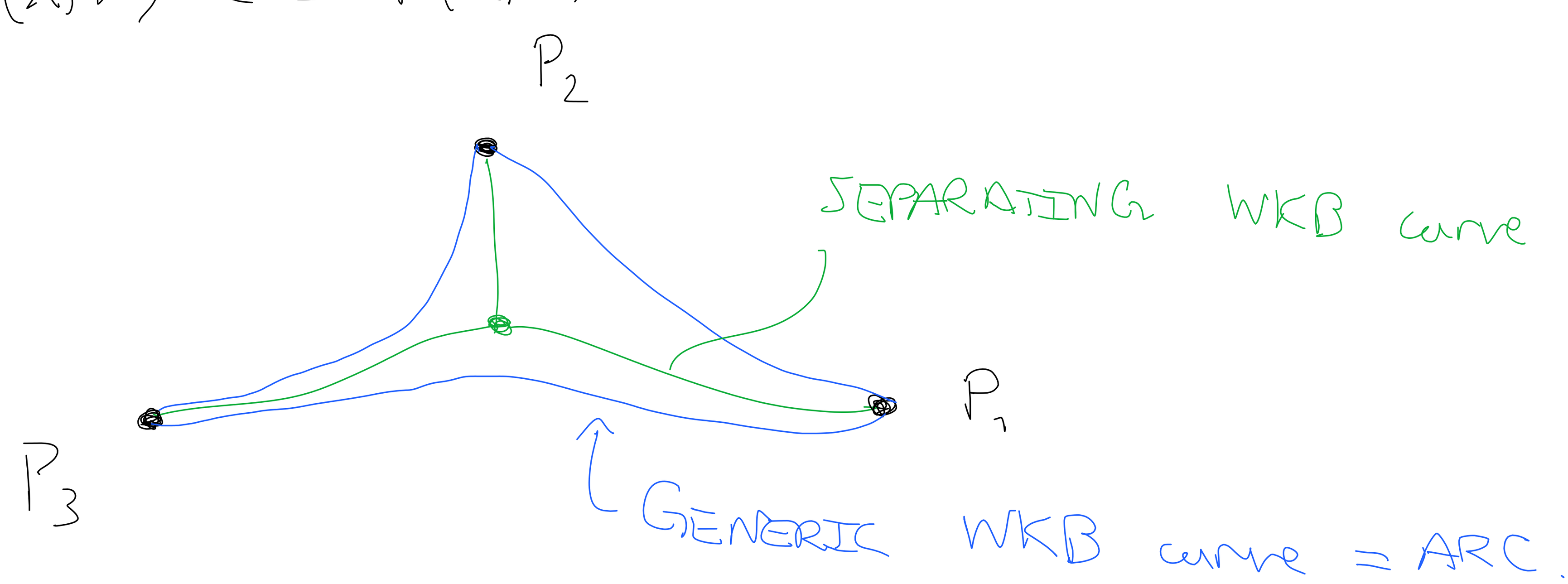
→ at a zero: (assume simple zeroes of $\frac{1}{2}\mathrm{tr}\,\varphi^2$) : Take $z(P_0) = 0$ and

$$\lambda = \left(\lambda_0\, z^{1/2} + \mathcal{O}(1) \right) dz$$

$$\Longrightarrow \quad z(t) = \left(C t + z_0 \right)^{2/3}$$



Associating : $(\lambda, \vartheta) \hookrightarrow T(u, \vartheta)$:



↳ WKB foliation / {separating curves} defines a continuous family of equivalent triangulations with representative $T(u, \vartheta)$.

5.2 : Labeling by homology:

GOAL: Associate $E \in T(u,\vartheta) \hookrightarrow \gamma \in \Gamma_u \cong \Gamma_u^e \oplus \Gamma_u^m$: BPS charge. ($\cong \mathbb{Z}^2$)

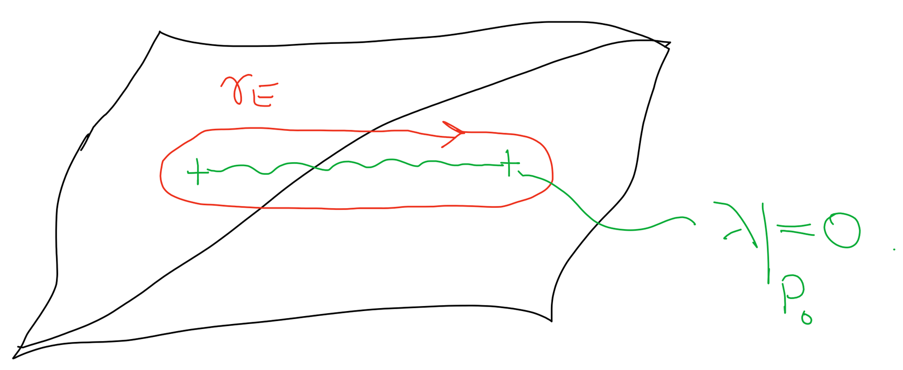
Define: Σ_u is the branched cover of C where λ is single-valued

↳ $\Gamma_u = H_1(\Sigma_u; \mathbb{Z})_{odd}$

transform with a sign under $\lambda \Leftrightarrow -\lambda$.

homology class of closed curves

Construction:

$Q_E =$ 

Ambiguity: orientation of γ_E and which branch of $\Sigma_u \to C$

↳ resolved by requiring $\langle \gamma_E, \gamma_{E'} \rangle = b_{EE'}$, $\forall E, E' \in T(u,\vartheta)$.

intersection pairing

5.3: BPS spectrum & wall-crossing:

Spectrum generator: There exists a unique decomposition

index that gives $\dim \mathcal{H}_{\gamma,u}$.

$$T(u,\vartheta+\pi) = \prod_\gamma K_\gamma^{\Omega_{\gamma,u}} \, T(u,\vartheta)$$

mutation at an edge or juggle at a cycle.

Wall crossing: $u \to u'$, $\prod_\gamma K_\gamma^{\Omega_{\gamma,u}} \longrightarrow \prod_\gamma K_\gamma^{\Omega_{\gamma,u'}}$

$$\{\Omega_{\gamma,u}\}_\gamma \neq \{\Omega_{\gamma,u'}\}_\gamma .$$