Y-systems from the thermodynamic Bethe ansatz

factorised scattering in 1+1 d

$\Downarrow$ (1)

thermodyn. Bethe ansatz (ground state density of particles)

$\Downarrow$ (2)

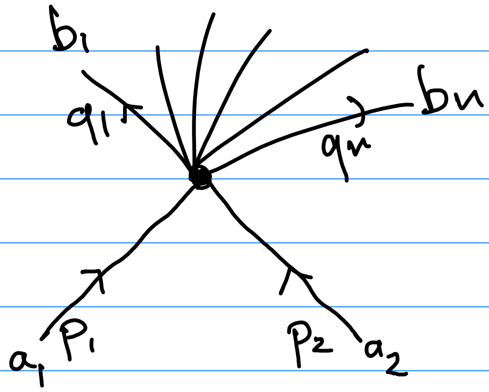
integral equations

$\Downarrow$ (3)

Y-systems

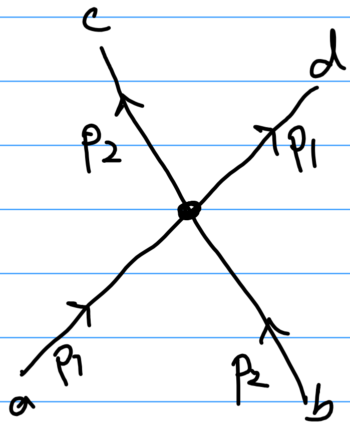
1) Factorised scattering

usual



factorised

- no $2 \to$ many

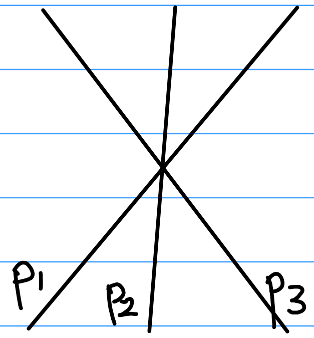


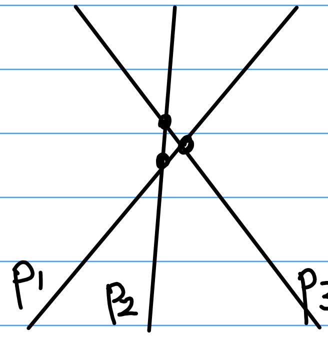
$$S_{ab}^{cd}(p_1, p_2)$$

one type only:

$$S(p_1, p_2) \in U(1)$$

- many → many
  as product





$$S_{3\to3} = S_{2\to2} \cdot S_{2\to2} \cdot S_{2\to2}$$

Comments:

- Connection to integrability "higher spin conserved charges"
- In d+1 with d>1 such theories are free
  (particles can avoid each other)

2) Thermodynamic Bethe ansatz

from:

Al.B. Zamolodchikov, "Thermodynamic Bethe ansatz in relativistic models: scaling 3-state Potts and Lee-Yang models."
Nuclear Physics B342 (1990) 695-720

S. J. van Tongeren, "Introduction to the thermodynamic Bethe ansatz"
J. Phys. A: Math. Theor. 49 (2016) 323005 , arXiv:1606.02951 [hep-th]

Rapidities: $p = (p^0, p^1)$, $p_0^2 - p_1^2 = m^2$

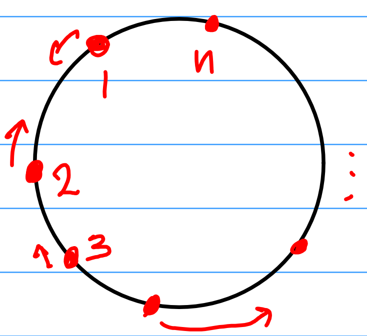
$$p^0 = m\cosh\beta\,,\quad p^1 = m\sinh\beta \quad \rightsquigarrow \text{ rapidity } \beta$$

Boost: $\Lambda_\gamma = \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix}$

$$\Lambda_\gamma\, p_\beta = p_{\beta+\gamma}$$

$$\rightsquigarrow S(p_{\beta_1}, p_{\beta_2}) = S(\beta_1 - \beta_2)$$

N particles on a circle, circumf. L



Bethe eqn:

$$e^{ip_iL}\prod_{j\neq i} S(\beta_i - \beta_j) = 1 \qquad i = 1, \dots, N$$

log

(*) $$mL\sinh\beta_i + \sum_{j\neq i}\delta(\beta_i - \beta_j) = 2\pi n_i$$

Thermodyn. limit : large L, adjacent levels β, β' :

$$\beta - \beta' \sim \frac{1}{L}$$

$\rightsquigarrow$ Particle density $\rho_1(\beta)$ $\sim L$

$$\rho_1(\beta)\,\Delta\beta = \#\left(\text{particles with rapidities in } [\beta, \beta+\Delta\beta]\right)$$

(*) becomes

$$m L \sinh \beta_i + \int_{-\infty}^{\infty} \delta(\beta_i - \beta')\,\rho_1(\beta')\,d\beta' = 2\pi n_i$$

Read as "Given the density ρ_1, what are the allowed rapidities of particles ?"

$$(**) \qquad \underbrace{m L \sinh \beta + \int_{-\infty}^{\infty} \delta(\beta - \beta')\,\rho_1(\beta')\,d\beta'}_{=:\, c(\beta)} = 2\pi n$$

$\Updownarrow$
Allowed states for a particle, given ρ_1
$\Updownarrow$
Density of states ρ

$$\rho(\beta)\,\Delta\beta = \#\left(\text{sol}^{\text{ns}} \text{ to } (**) \text{ in } [\beta, \beta+\Delta\beta]\right)$$

(for fixed ρ_1)

$$= \frac{1}{2\pi}\frac{d}{d\beta} c(\beta) \cdot \Delta\beta$$

Self consistency eqn. for $\rho(\beta)$: total density of states
$\rho_1(\beta)$: density of occupied states

$$(***) \qquad 2\pi\, \rho(\beta) = mL \cosh\beta + \int_{-\infty}^{\infty} \underbrace{\varphi(\beta-\beta')}_{\varphi = \frac{\partial \delta}{\partial \beta}} \rho_1(\beta')\, d\beta'$$

<u>Energy</u>

$$E(\rho_1) = \int_{-\infty}^{\infty} m \cosh\beta \cdot \rho_1(\beta)\, d\beta$$

<u>Entropy</u>

In rapidity interval $[\beta, \beta+\Delta\beta]$:

N states $\qquad N = \rho(\beta)\Delta\beta$
n particles $\qquad n = \rho_1(\beta)\Delta\beta$

Assume bosonic particle.

$$M_\beta = \frac{(N+n-1)!}{(N-1)!\, n!}$$

e.g. $n=2$ particles in $N=3$ states

etc : 3

etc : 3

$$\frac{(3+2-1)!}{(3-1)!\,2!} = \frac{4!}{2!\,2!} = \frac{24}{4} = 6$$

Entropy $S(\rho,\rho_1) \approx \sum_\beta \ln M_\beta$

Stirling & L large : $\ln(m!) \sim m \ln m$

$$= \int d\beta \left((\rho+\rho_1)\ln(\rho+\rho_1) - \rho \ln \rho - \rho_1 \ln \rho_1 \right)$$

Free energy

$$F = E - TS$$

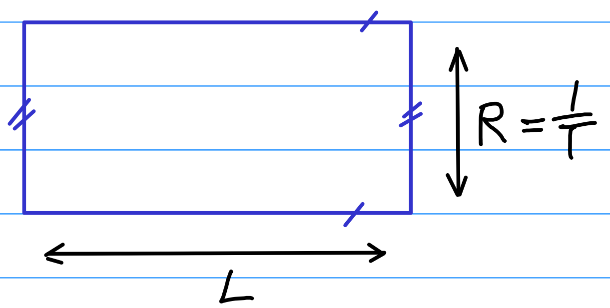
$$F(\rho,\rho_1) = E(\rho_1) - TS(\rho,\rho_1)$$

Minimise F : $F(\rho+\delta\rho, \rho_1+\delta\rho_1)$

- from (***) :

$$2\pi\,\delta\rho = \int_{-\infty}^{\infty} \varphi(\beta-\beta')\,\delta\rho_1(\beta')\,d\beta'$$

- write $\frac{1}{T} = R$:



(1+1 dim quantum system at temp. T
= 2d statistical system on torus of circumf. $R = \frac{1}{T}$)

- write $e^{\varepsilon} = \frac{\rho+\rho_1}{\rho_1}$ "pseudoenergy"

$$E(\rho_1+\delta\rho_1) = E(\rho_1) + \int_{-\infty}^{\infty} m\cosh\beta \cdot \delta\rho_1(\beta)\, d\beta$$

$$(f+\delta f)\ln(f+\delta f) = \delta f\cdot \ln f + f\left(\ln f + \frac{\delta f}{f}\right) = f\ln f + \delta f(1+\ln f)$$

$$S(\rho+\delta\rho, \rho_1+\delta\rho_1) = S(\rho,\rho_1) + \int_{-\infty}^{\infty} d\beta \Big(\underbrace{(\delta\rho+\delta\rho_1)(\cancel{1}+\ln(\rho+\rho_1)) - \delta\rho(\cancel{1}+\ln\rho) - \delta\rho_1(\cancel{1}+\ln\rho_1)}_{= \delta\rho\cdot\ln\frac{\rho+\rho_1}{\rho} + \delta\rho_1 \ln\frac{\rho+\rho_1}{\rho_1}} \Big)$$

use $\displaystyle\int_{-\infty}^{\infty} d\beta\, \delta\rho(\beta) \underbrace{\ln\left(1+\frac{\rho_1}{\rho}\right)}_{=h(\beta)} = \int_{-\infty}^{\infty} d\beta\, d\beta' \frac{1}{2\pi}\varphi(\beta-\beta')\delta\rho_1(\beta')\, h(\beta) \underset{\beta\leftrightarrow\beta'}{=} \int d\beta\, \delta\rho_1(\beta) \int d\beta' \frac{1}{2\pi}\varphi(\beta'-\beta) h(\beta')$

$$S(\rho+\delta\rho, \rho_1+\delta\rho_1) = S(\rho,\rho_1) + \int d\beta\, \delta\rho_1(\beta)\left(\ln\left(1+\frac{\rho}{\rho_1}\right) + \frac{1}{2\pi}\int d\beta'\, \varphi(\beta'-\beta) h(\beta')\right)$$

$$Rm\cosh\beta - \ln\underbrace{\frac{\rho+\rho_1}{\rho_1}(\beta)}_{=e^{\varepsilon(\beta)}} - \frac{1}{2\pi}\int d\beta'\, \varphi(\beta-\beta') \ln\underbrace{\frac{\rho+\rho_1}{\rho}(\beta')}_{=\left(\frac{\rho}{\rho+\rho_1}\right)^{-1} = \left(1-\frac{\rho_1}{\rho+\rho_1}\right)^{-1} = (1-e^{-\varepsilon})^{-1}}$$

Vanishing variation:

$$\boxed{-Rm\cosh\beta + \varepsilon(\beta) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\beta-\beta') \ln\left(1-e^{-\varepsilon(\beta')}\right) d\beta' = 0}$$

With $\varepsilon(\beta)$:

- get free energy
- $R\to 0$ gives central charge of UV CFT

3) Y-systems

from

A. Kuniba, T. Nakanishi, J. Suzuki, "T-systems and Y-systems in integrable systems"
J.Phys.A44:103001,2011 , arXiv:1010.1344 [hep-th]

The unrestricted Y-system for $\mathfrak{g}$ is the following relations among commuting variables $\{Y_m^{(a)}(u) \mid a \in I, m \in \mathbb{Z}_{\geq 1}, u \in U\}$, where $Y_m^{(0)}(u) = Y_0^{(a)}(u)^{-1} = 0$ if they occur in the RHS.

For simply laced $\mathfrak{g}$,

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{\prod_{b\in I:C_{ab}=-1}(1+Y_m^{(b)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}. \tag{2.11}$$

Restricted Y-system of level $\ell \geq 2$: $1 \leq m < \ell$, $Y_\ell^{(a)}(u)^{-1} = 0$
(simply laced case)

Theorem 5.6 (Periodicity [120, 121, 122, 123, 124, 115, 125, 17, 94, 96]). *For any family of variables* $\{Y_m^{(a)}(u) \mid a \in I, 1 \leq m \leq t_a\ell - 1, u \in \mathbb{Z}\}$ *satisfying the level* ℓ *restricted Y-system for* $\mathfrak{g}$, *one has the periodicity*

$$Y_m^{(a)}(u + 2(h^\vee + \ell)) = Y_m^{(a)}(u). \tag{5.26}$$

To prove Theorem 5.6 in full generality, the use of the categorification of the cluster algebra by the cluster category by [117, 118] is essential.

from

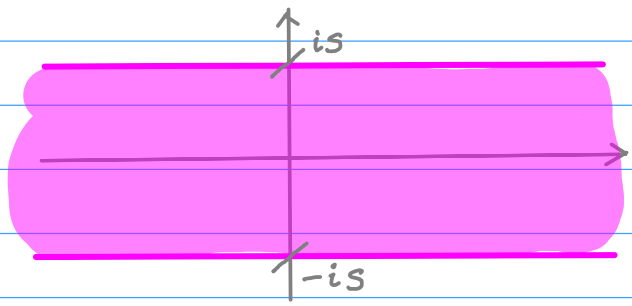
L. Hilfiker, I. Runkel, "Existence and uniqueness of solutions to Y-systems and TBA equations", Annales Henri Poincaré 21, 941-991 (2020) , arXiv:1708.00001 [math-ph]

Fix $N \in \mathbb{Z}_{>0}$ $\quad s \in \mathbb{R}_{>0}$

$G \in \mathrm{Mat}_N(\mathbb{R})$ s.th.
- diagonalisable
- eval. $\in (-2,2)$
- non-negative $(G_{nm} \geq 0)$
- irreducible

(cannot permute to block upper-triangular form)

e.g.:
- $N=1$, $G=(1)$
- G adjacency matrix of finite Dynkin diag. of rank N

$\bar{S}$ = 

(closed strip in $\mathbb{C}$)

A_s = { continuous fn $f : \bar{S} \to \mathbb{C}$, analytic in interior of $\bar{S}$ }

Y-system: $Y_1, \dots, Y_N \in A_s$, $n = 1, \dots, N$, $x \in \mathbb{R}$

(*) $$\boxed{Y_n(x+is)\, Y_n(x-is) = \prod_{m=1}^{N} \left(1 + Y_m(x)\right)^{G_{nm}}}$$

Relation to above for:

- level $\ell = 2$
- G: adjacency matrix of simply laced Lie alg.

Two choices of functions $a_n \in \mathcal{A}_s$, $n=1,\dots,N$:

1) $a_n = 0$

2) Let w: Perron–Frobenius eigenvector of G

$$G w = 2\cos(\gamma)\, w$$

$$a_n(x) = r\, w_n \cosh\left(\frac{\gamma x}{s}\right) \qquad (r>0)$$

<u>Thm 1</u>: $\boxed{\exists!}$ soln $Y_1, \dots, Y_N$ to (*) with properties

1) $Y_n(\mathbb{R}) \subset \mathbb{R}_{>0}$

2) $Y_n(z) \neq 0$ for all $z \in \overline{S}$

3) $\log Y_n(z) - a_n(z)$ is bounded on $\overline{S}$

<u>Detour</u>: constant Y-systems

<u>Lem.</u>: If $a_1, \dots, a_N = 0$, then the unique soln $Y_1, \dots, Y_N$ to (*) consists of constant functions.

<u>Cor.</u>: The set of equations

$$Y_n^{\,2} = \prod_{m=1}^{N} (1+Y_m)^{G_{nm}} \qquad (n=1,\dots,N)$$

has a unique positive solution.

4) Integral equations

Set

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\left(2\cosh(sk)\,\mathbb{1}_{N\times N} - G\right)^{-1} dk$$

For $f_1, \dots, f_N : \mathbb{R} \to \mathbb{R}$ bounded & continuous :

(**) $$f_n(x) = \sum_{m=1}^{N} \int_{-\infty}^{\infty} \left[\phi(x-y)G\right]_{nm} \cdot \ln\left(1 + e^{-f_m(y) - a_m(y)}\right) dy$$

Compare to TBA eqn from before:

$$\varepsilon(\beta) - Rm\cosh\beta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\beta-\beta')\ln\left(1 - e^{-\varepsilon(\beta')}\right) d\beta'$$

same for $N=1$, $G=1$, $\varepsilon(\beta) = f(\beta) + a(\beta)$

with $a(\beta) = Rm\cosh\beta$

Thm. 2:

(i) $\exists!$ bounded & cont. sol$^{\underline{n}}$ $f_1, \dots, f_n$ to $(**)$.

(ii) This sol$^{\underline{n}}$ extends to functions $f_1, \dots, f_n \in \mathcal{A}_s$, bounded on $\overline{S}$, and

$$Y_n(z) = e^{a_n(z) + f_n(z)}$$

is the unique sol$^{\underline{n}}$ to the Y-system $(*)$.

Comparison to before:

$$Y(z) = e^{\varepsilon(z)}$$

$$Y(x+is)\,Y(x-is) = 1 + Y(x)$$

(and $s = \frac{\pi}{3}$ since in $G w = 2\cos\gamma\, w$ have $\cos\gamma = \frac{1}{2}$,
i.e. $\gamma = \pi/3$ and $a(x) = r w \cosh(\gamma x/s)$, so need $s = \gamma$]

Main steps in proof of Thm 1 & 2:

- Rewrite (**) & use Banach fixed point thm. to get Thm 2 (i)

- Take log of (*)

$$f_n(x+is) + f_n(x-is)$$

$$= \sum_{m=1}^{N} G_{nm} \log\left(e^{-a_m(y)} + e^{f_m(y)}\right)$$

and write as

$$f_n(x+is) + f_n(x-is) - (Gf)_n(x) = g_n(x)$$

- Dirac δ-distribution

$$\phi(x+is) + \phi(x-is) - G\phi(x) = \delta(x)\, \mathbb{1}_{N\times N}$$

gives

$$f_n(x) = \int_{-\infty}^{\infty} \sum_m \phi_{nm}(x-y)\, g_m(y)\, dy$$

$\Leftrightarrow$ (**)

- Gives Thm 2 (ii) & Thm 1.