QFT I exercises - sheet 4

Solutions to Dirac

Consider

$$\psi^{+} = u^s(p)e^{-\mathrm{i}p\cdot x} \quad \text{with} \quad u^s(p) = \frac{1}{N}\begin{pmatrix} (p\cdot\sigma + m)\xi^s \\ (p\cdot\bar{\sigma} + m)\xi^s \end{pmatrix} \tag{1}$$

$$\psi^{-} = v^s(p)e^{\mathrm{i}p\cdot x} \quad \text{with} \quad v^s(p) = \frac{1}{N}\begin{pmatrix} (p\cdot\sigma + m)\xi^s \\ -(p\cdot\bar{\sigma} + m)\xi^s \end{pmatrix} \tag{2}$$

with

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

a Under which condition are $\psi^{\pm}(p)$ a solution to the Dirac equation? Tip: if all else fails, first go to the rest frame.

b Verify *in the rest frame* that the index on ξ corresponds to eigenvalues of the last component of the Pauli-Lubanski vector, $\hat{W}^3$ and identify the quantum number.

For the spinors $u(p)$ $v(p)$ a normalisation N exits for which

$$\bar{u}^s u^r = 2m\delta^{rs} \quad \bar{v}^s v^r = -2m\delta^{rs} \quad \bar{u}^s v^r = 0 = \bar{v}^s u^r \tag{4}$$

$$(u^s)^{\dagger}u^r = 2p^0\delta^{rs} \quad (v^s)^{\dagger}v^r = 2p^0\delta^{rs} \tag{5}$$

c Find the normalisation and prove the equations.

d Show that for this normalisation $\sum_r u^r\bar{u}^r = \gamma_\mu p^\mu + m$ as well as $\sum_r v^r\bar{v}^r = \gamma_\mu p^\mu - m$ hold.

e Show the Gordon-identity

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}(p')(p'^\mu + p^\mu + 2\mathrm{i}J_S^{\mu\nu}(p'-p)_\nu)u(p) \tag{6}$$

where $(\gamma^\mu p_\mu - m)u(p) = 0 = \bar{u}(p')(\gamma^\mu p'_\mu - m)$ and $J_S^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$

Quantization: right and wrong

Someone writes a Dirac field in the Heisenberg picture as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a^s_{\vec{p}} u^s e^{-\mathrm{i}p\cdot x} + b^{\dagger s}_{\vec{p}} v^s e^{\mathrm{i}p\cdot x} \right) \tag{7}$$

a What is missing? Hint: it is almost trivial...

b Write the natural expansion for $\bar{\psi}$

c Derive the Hamilton and spatial momentum classical Noether charges from the Dirac Lagrangian.

d Show the classical Noether charge for the free Dirac Lagrangian for the symmetry $\psi \to e^{\mathrm{i}\alpha}\psi$ is $Q = \int d^3x \psi^\dagger \psi$

Consider quantisation by promoting a and b to operators which obey $[a^r_{\vec{p}}, a^{s,\dagger}_{\vec{p}'}] = \delta^{rs}(2\pi)^3 \delta^3(p - p')$

e Derive the Hamilton operator using these relations.

f Repeat e), but now for anti-commutation relations

g Repeat f) for the spatial momentum operator, P^i.

h Compute the operator corresponding to the charge in d) in the quantum theory using anti-commutation relations

i When is $\psi \to e^{\mathrm{i}\alpha\gamma_5}\psi$ a symmetry of the Dirac Lagrangian?