QFT I exercises - sheet 3

To be handed in 6th of November @ start of lecture (12:00). Exercises labelled "BONUS" are not counted towards the result: they are meant as a challenge :).

Green's functions, schmeen's functions

Let ϕ be a free real scalar field. It's Lagrangian is, as usual, given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \tag{1}$$

This field can be canonically quantised as discussed in the lecture (and Peshkin&Schroeder). Below ϕ is always the 'quantised' *operator-valued* field.

- a Compute $\langle 0|\phi(x)\phi(y)|0\rangle$.
- b Argue translation invariance dictates that the answer should not depend on $x^\mu + y^\mu$ as long as the vacuum is invariant under translations. Hint: $\mathcal{O}(x+a) = e^{iaP}\mathcal{O}(x)e^{-iaP}$ for any operator $\mathcal{O}$ where P generates translations.
- c Compute $\langle 0|[\phi(x), \phi(y)]|0\rangle$. Hint: recycle a)!
- d Compute $\langle 0|T(\phi(x)\phi(y))|0\rangle$ which is defined as $\theta(x_0 - y_0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y_0 - x_0)\langle 0|\phi(x)\phi(y)|0\rangle$. Hint: recycle a)!
- e Compute $\langle 0|\phi(x)\phi(y)\phi(z)|0\rangle$.
- f Why is the vacuum expectation value of an odd number of ϕ's ($\sim \langle 0|\phi^{\mathrm{odd}}|0\rangle$) always zero?
- g BONUS: Show $\langle 0|\phi(w)\phi(x)\phi(y)\phi(z)|0\rangle = D(w-x)D(y-z) + D(w-y)D(x-z) + D(w-z)D(x-y)$ where $D(x-y)$ is the answer to a). Guess the structure of the result for the vacuum expectation value of six and more fields as a sum over products of D functions.

The vacuum expectation values just computed have an interpretation as a Green's function for the Klein-Gordon equation. That is, the formal solution to

$$(\partial^2 + m^2)G(x-y) = -i\delta^4(x-y) \tag{2}$$

- h Show the Green's function in momentum space is $G(p) = \frac{\mathrm{i}}{p^2 - m^2}$.
- i Write down the formal expression for $G(x-y)$. Why is this 'formal' only?

j Interpret the integral along p^0 as a contour integral in the complex plane. Show explicitly which contour leads to the answer of question d).

k Show this leads to $\langle 0|T(\phi(x)\phi(y))|0\rangle = \lim_{\epsilon\downarrow 0}\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-\mathrm{i}p(x-y)}$

Poincare algebra

Let $\hat{J}_{\mu\nu}$ be the operator

$$\hat{J}_{\mu\nu} = \mathrm{i}\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \tag{3}$$

and

$$\hat{P}^\mu = \mathrm{i}\partial^\mu \tag{4}$$

a Compute the commutator $[\hat{J}^{\mu\nu}, \hat{J}^{\rho\sigma}]$ to show these operators satisfy the Lorentz algebra.

b Show $\hat{P}^\mu$ is the generator of translations. Hint: consider the effect of a translation on a function $\phi(x)$

c Compute $[\hat{P}^\mu, \hat{P}^\nu]$

d Compute $[\hat{P}^\mu, \hat{J}^{\nu\rho}]$. Hint: you could guess the form of the answer.

e Show that for infinitesimal rotations parametrised by a set of three angles θ_i one has

$$\delta x^i = \frac{\mathrm{i}}{2}\theta_k\epsilon^{kmn}\hat{J}_{mn}x^i \qquad \text{with } \{i,k,m,n\} \in \{1,2,3\} \tag{5}$$

by explicit computation and comparison to exercise sheet 0.

f Obtain the corresponding formula for the change in x^μ under infinitesimal boosts parameterised by three velocities v_i

The algebra obeyed by the generators $\hat{J}$ and $\hat{P}$ in this exercise is known as the Poincare algebra.

g Define a new generator $\hat{\Delta} = \mathrm{i}x^\mu\partial_\mu$. Which transformation does this generator generate? Hint: compute $(e^{\mathrm{i}\lambda\hat{\Delta}})x$

h Compute the commutator of $\hat{\Delta}$ with $\hat{J}$, $\hat{P}$ and itself.

i BONUS: argue that $\hat{\Delta}$ cannot correspond to the generator of an exact symmetry of nature.