QFT I exercises - sheet 2

Free scalar quantum fields

For a free real scalar field of mass m we have the Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \tag{1}$$

We can introduce new coordinates

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}} + a^\dagger_{-\vec{p}}\right)e^{\mathrm{i}\vec{p}\vec{x}} \tag{2}$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}(-i)\left(a_{\vec{p}} - a^\dagger_{-\vec{p}}\right)e^{\mathrm{i}\vec{p}\vec{x}} \tag{3}$$

After quantisation the a and $a^\dagger$ are operators. Moreover, $E_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$.

a Calculate the Hamiltonian operator (Hint: guess it's form first!)

The momentum operator follows from the energy-momentum tensor as the Noether charge associated to translations.

b Calculate the momentum operator P^i

c Compute energy and momentum of the state $a^\dagger_{\vec{q}}|0\rangle$ where $\vec{q}$ is some three vector and suggest an interpretation of this state

Now we generalize to the complex scalar field with Lagrangian,

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi \tag{4}$$

e Write the fields ϕ and ϕ^* and their momentum densities π and π^* in terms of creation and annihilation operators as above. How are the creation and annihilation operators of the complex fields related?

f Suggest a change of coordinates which changes this Lagrangian into a sum over two Lagrangians for real scalar fields.

g Repeat exercise b) and c) for the complex scalar field in these new coordinates.

h Derive the commutation relations of the complex fields ϕ, ϕ^*, π, π^*.

As discussed last week, the complex scalar field has a phase rotation symmetry for $\phi \to e^{\mathrm{i}\alpha}\phi$. By Noether's theorem this leads to the existence of a conserved Noether charge Q_N

i Guess what $Q_N\phi(x)|0\rangle$ and $Q_N\phi^*(x)|0\rangle$ will evaluate to.

- j Compute the operator Q_N and the commutation relations $[Q_N, \phi(x)]$ and $[Q_N, \phi^*(x)]$.
- k Compute $Q_N\phi(x)|0\rangle$ and $Q_N\phi^*(x)|0\rangle$. Hint: compute $Q_N|0\rangle$ first.
- l Suggest an interpretation of $\phi(x)|0\rangle$
- m Compute $\langle 0|\phi(x)\phi(y)|0\rangle$, $\langle 0|\phi^*(x)\phi^*(y)|0\rangle$, $\langle 0|\phi(x)\phi^*(y)|0\rangle$ and $\langle 0|\phi^*(x)\phi(y)|0\rangle$

More Klein-Gordon fun

For a free real scalar field of mass m we have the Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \tag{5}$$

Consider now the $4D$ Fourier transform,

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4}\phi(p)e^{\mathrm{i}p_\mu x^\mu} \tag{6}$$

- a Show the solution to the Klein-Gordon equation can be written as $\phi(p) = \frac{a_{\vec{p}}}{\sqrt{2E_p}}\delta(p^0 - E_p) + \frac{a^\dagger_{\vec{p}}}{\sqrt{2E_p}}\delta(p^0 + E_p)$

Now add a term to the Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + J\phi \tag{7}$$

with a new field $J(x)$. This field will be treated as a classical field.

- b Derive the equation of motion and write the most general solution in terms of the retarded Greens function.
- c Introduce a new field $\tilde{\phi}$ and $\tilde{J}$ such that the Lagrangian does not contain $\tilde{\phi}\tilde{J}$-type terms. (Hint: complete the square)
- d $J(y)$ is switched on and off in the past ($y^0 < x^0$). Show the solution to the equation of motion $\phi(x)$ can be written as a), with redefined coefficients $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$
- e Compute $\langle 0|H|0\rangle$ after j has been switched off.
- f Interpret the answer of d). Why is J sometimes called a source?