## QFT I exercises - sheet 11

The following is meant as an illustration of the power of simple physical concepts such as Lorentz invariance and unitarity. However, it is also meant as a bonus for those that like a challenge. The results of this exercise are therefore not part of the exam.

## A poor man's no-ghost theorem

In this exercise it is shown that the so-called Veneziano amplitude,

$$
A_{4}(s, t)=\frac{\Gamma\left(-\alpha^{\prime} s-\alpha_{0}\right) \Gamma\left(-\alpha^{\prime} t-\alpha_{0}\right)}{\Gamma\left(-\alpha^{\prime}(t+s)-2 \alpha_{0}\right)}
$$

can only be interpreted as part of a unitary quantum field theory if the number of dimensions obeys $D \leq 26$ and the 'intercept' $\alpha_{0}$ obeys $\alpha_{0} \leq 11^{1}$. The Veneziano amplitude is a scattering amplitude in bosonic string theory. It has it's own wikipedia page in english.
a. What is the mass dimension of $\alpha^{\prime}$ ? Guess its natural size. Hint: what kind of theory is string theory supposed to be?
b. Locate the poles in the Veneziano amplitude in the $s=-\left(k_{1}+k_{2}\right)^{2}$ and $t=-\left(k_{2}+k_{3}\right)^{2}$ channel. (Bonus: locate the zeros as well.)
c. Convince yourself from what I said in the lecture that

$$
\lim _{\alpha^{\prime} s \rightarrow X}\left(\alpha^{\prime} s-X\right) A_{4}(s, t)=\sum_{\text {all states }} A_{3}^{L} A_{3}^{R}
$$

should hold, where $X$ is the location of one of the poles found in $b$. Moreover, from the explicit expression for the amplitude show,

$$
\lim _{\alpha^{\prime} s \rightarrow X}\left(\alpha^{\prime} s-X\right) A_{4}(s, t)=\frac{1}{n!} \prod_{i=0}^{n}\left(\alpha^{\prime} t+\alpha_{0}+i\right)
$$

with $n$ related to $X$.
d. The external particles on the Veneziano amplitude must be scalars, of mass $m$ which we will label ' T '. Argue that the three point amplitudes in $c$ must be of the form

$$
A_{3}^{R}\left(T, T, M^{(j)}\right) \propto \prod_{i}^{j} \sqrt{\frac{\alpha^{\prime}}{2}}\left(k_{1}-k_{2}\right)^{\mu_{i}} \xi_{\mu_{i}}^{I_{i}}
$$

for some integer $j$ by Lorentz invariance and momentum conservation (take all momenta to be inward pointing). Here $\xi_{\mu}^{I}$ is the polarization

[^0]vector of a massive or massless vector boson, depending on the particle $M$ and $I$ runs from 1 to $D-1$ in the massive and $D-2$ in the massless case. Hint: the polarisation vector must be transverse and terms such as $\propto \delta^{I_{1} I_{2}}$ can be ignored.
e. Bonus: argue that $j$ is the (maximal) spin in $D=4$.
f. Show that a particular contribution to the right hand side in question $c$ must be
$\sum_{\text {all states with a fixed } j} A_{3}^{R}\left(T, T, M^{(j)}\right) A_{3}^{R}\left(T, T, M^{(j)}\right)=c_{j}^{2}\left(\alpha^{\prime} t-2 \alpha^{\prime} m^{2}+\frac{X}{2}\right)^{j}$
where the sum runs over all indices of the polarisation vectors and $c_{j}$ is a proportionality factor. (Hint: start with $j=1$, and use that $\sum_{I} \xi_{I}^{\mu} \xi_{I}^{\nu}=\eta^{\mu \nu}+\ldots$ where the dots do not matter for our calculation
g. Argue that $c_{j}$ must be real. Hint: this amplitude would follow from a single Feynman graph. What constraint must hold for the interaction Hamiltonian?
h. Interpreting the lowest mass pole of the amplitude as the same particle type as those on the 'outside', show this fixes $m^{2} \alpha^{\prime}=-1 / \alpha_{0}$.
i. From the next-to-lowest mass pole, obtain the fact that there is a single massless boson at this level. Furthermore, show $\alpha_{0} \leq 1$ if the proportionality constants $c_{j}$ are real. (it will turn out for more complicated reasons that $\alpha_{0}=1$ which we take from here on)
j. From the next-to-next-to lowest mass pole, obtain $D \leq 26$ if the proportionality constants $c_{j}$ are real. Crucial hint: compute the contribution for a particle which has a polarisation tensor $\propto \xi_{\mu}^{I} \xi_{\nu}^{J}$ which is traceless in $I$ and $J$ : this brings in the dimension. Show then that you cannot add other particles to the mix without violating $c_{j}$ is real if $D=26$.
k. Bonus 1: based on the fact that the external particles are bosons, argue that the amplitude as given above is in this form incomplete. Obtain the most generic completion you can think of.
l. Bonus 2: show the Dynkin labels of the states coupling in amplitudes of the type obtained in $d$ are $(\lambda, 0,0, \ldots)$. Compare to equation 3.27 in 1007.2622 [hep-th]. Do all states couple to two tachyons?

The constraints on $D$ and $\alpha_{0}$ are part of something known as the 'no ghost' theorem in bosonic string theory. There is proof that if these constraints hold every scattering amplitude in the theory is unitary.


[^0]:    ${ }^{1}$ See also P. Frampton, Phys.Lett. B41 (1972) 364-370

