QFT I exercises - sheet 1

Fun with Noether

Consider a complex scalar field ϕ with Lagrangian density

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

- a Compute the equation of motion for ϕ with the Euler-Lagrange equation.
- b Compute the energy-momentum tensor $T_{\mu\nu}$ and check $\partial_\mu T^{\mu\nu} = 0$.
- c Compute the momentum density $\pi(x)$ and $\pi^*(x)$ conjugated to ϕ^* and ϕ.
- d Compute the Hamilton density from the previous result and verify explicitly that $T_{00} = \mathcal{H}$
- e Verify the Lagrangian is invariant under $\phi \to e^{\mathrm{i}\alpha}\phi$ for a constant α
- f Compute the Noether current j^μ associated to this transformation.
- g Compute the variation of the Lagrangian under $\phi \to e^{\mathrm{i}\alpha}\phi$ for a non-constant α. How is this result related to the Noether current?

We add a term $\propto A_\mu j^\mu$ to the Lagrangian with a new field A_μ. Consider transforming $\phi \to e^{\mathrm{i}\alpha}\phi$ together with $A_\mu \to A_\mu + \delta_\mu$ for some vector δ_μ.

- h What should δ be to make the new Lagrangian invariant under this transformation for constant α?
- i What should δ be to make the new Lagrangian invariant under this transformation for *non*-constant $\alpha(x)$ (disregarding the transformation of j in $j_\mu A^\mu$)?
- j Suggest/guess a physical interpretation of A_μ.
- k (BONUS) Consider the variation of the current j in the $j_\mu A^\mu$ term. Find the term needed to make the Lagrangian completely invariant.
- l (BONUS) The Lagrangian given above is invariant under constant translations. Give an argument (not necessarily sufficient :)) that to make the Lagrangian invariant under non-constant translations one has to introduce a field with *two* Lorentz indices. Suggest/guess a physical interpretation of said field.

The exercise above contains a number of steps to make a Lagrangian which is invariant under a constant ('global') symmetry transformation invariant under non-constant ('local') transformation. This is sometimes known as the 'Noether procedure'.

The Lagrangian of Electrodynamics

Let $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ be the electromagnetic field strength tensor. Consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

where j_μ is some real function.

a Show the first term in $\mathcal{L}$ is invariant under $A_\mu \to A_\mu + \partial_\mu f(x)$ for an arbitrary function $f(x)$

b Show the Euler-Lagrange equations for this Lagrangian are equivalent to the Maxwell equations

There are more terms we could add to the Lagrangian which are Lorentz invariant and invariant under $A_\mu \to A_\mu + \partial_\mu f(x)$. An example of such a term is

$$\propto F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$$

c Show that $F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = \partial_\mu C^\mu$ for some vector C^μ. Hint: first show that anything containing $\ldots \epsilon^{\mu\nu\rho\sigma} (\partial_\mu \partial_\nu f(x))$ is zero.

d Argue that because of this $F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$ does not change under local variations of the field A which go to zero fast enough at infinity.

e Argue that because of this $F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$ will not contribute to the equations of motion of A^μ

The vector C^μ is a special case of what is known as a 'Chern-Simons current'. The particular combination $F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$ is known in physics as a 'topological invariant'.