QFT I exercises - sheet 0

Rotations

Vectors in three dimensions can be written in a Cartesian coordinate system as columns of three numbers, $\mathbf{v}$. A rotation along any of the axis is given by a matrix D acting on this.

a Show $\mathbf{v}^T\mathbf{v}$ is the Cartesian length of the vector

b Rotations keep the Cartesian length invariant by definition. Show this means $D^TD = 1$. Derive the same using index notation.

c Construct the rotation matrix D for a rotation in the x-y plane along an angle α_1 such that $\det D = 1$ and $D(\alpha_1 = 0) = 1$.

d Which form does a generic rotation matrix have with rotation angles $\alpha_1, \alpha_2, \alpha_3$?

e Expand D to first order in the rotation angles and determine the 3x3 matrices T^i in

$$D = 1 + \mathrm{i}\sum_{i=1}^{3} \alpha_i T^i + \mathcal{O}(\alpha^2)$$

f Compute $[T^i, T^j]$. What does this commutator measure?

g Lorentz transformations can be defined as the transformations Λ which leave $\eta = \mathrm{Diag}(1, -1, -1, -1)$ invariant. Show that rotations in $\mathbb{R}^3$ are a special case of Lorentz transformations.

h Show that

$$\Lambda = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a Lorentz transformation

i Show that Lorentz transformations satisfy $\det\Lambda = \pm 1$

j What is the shortest distance between two points when using the Minkowski metric η to measure distance? Draw a path in a lightcone diagram .

Electrodynamics

Let $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ be the electromagnetic field strength tensor.

a Show F is invariant under $A_\mu \to A_\mu + \partial_\mu f(x)$ for an arbitrary function $f(x)$

b Show F is a tensor if A is.

c Show the equations $\partial^\mu F_{\mu\nu} = j_\nu$ are equivalent to the Maxwell equations if j_ν is the covariant version of the electric current density.

d Show the equation $\partial^\mu F_{\mu\nu} = j_\nu$ is covariant under Lorentz transformations.

e Define a new tensor $*F_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. Show this operation interchanges E and B.

f Show $\partial_\mu * F^{\mu\nu} = 0$ if F is given as above in terms of A.

f If there were magnetic monopoles, what would their 'Maxwell' equation be?

h Argue that for magnetic monopoles to exist the description in terms of a potential A should break down somewhere.

Further reading: "magnetic monopole" on Wikipedia and especially Paul Dirac, "Quantised Singularities in the Electromagnetic Field". Proc. Roy. Soc. (London) A 133, 60 (1931). This article can be found on the web, e.g. through the wikipedia article.