Statistical Error of Efficiency Determination from Weighted Events

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1 Prerequisites

It is well known that the statistical error of the efficiency ϵ determined from a number of n events as

$$\epsilon = \frac{n_+}{n},$$

where n_+ is the number of events in the sample that fulfill a certain condition (e. g. have been triggered), is given by

$$\delta\epsilon = \sqrt{\frac{n_+ n_-}{n^3}} = \sqrt{\frac{\epsilon\,(1-\epsilon)}{n}}, \tag{1}$$

where $n_- = n - n_+$ is the number of events that do not fulfill the given condition.

This formula may be derived from assuming n_+ and n_- to be individually and independently poissonian distributed, therefore having errors

$$\delta n_\pm = \sqrt{n_\pm},$$

where $\pm$ stands for either $+$ or $-$.

This is completely equivalent to the approach to assume n to be fixed, and n_+ being binomially distributed according to

$$\mathcal{P}\,(n_+, n) = \left(\begin{array}{c} n \\ n_+ \end{array} \right) p^{n_+} (1-p)^{n-n_+},$$

where p is the true efficiency, taking the measured ϵ as estimate for p and using the known standard deviation of a binomially distributed quantity $np(1-p) = n_+ n_- / n^2$ for the error calculation.

The reason for this equivalency is the following: Take two quantities n_+ and n_- that are independently distributed according to a poisson distribution

$$\mathcal{P}_\pm\,(n_\pm) = \exp\,(-\lambda) \frac{\lambda_\pm^{n_\pm}}{n_\pm!},$$

where λ_+ and λ_- are the mean values of n_+ and n_-, respectively. If the sum $n = n_+ + n_-$ is fixed (i. e. one looks at a diagonal slice in the (n_+, n_-) plane), then n_+ in this slice is distributed according to a binomial distribution with the parameter $p = \lambda_+/(\lambda_+ + \lambda_-)$, as can easily be proven.

2 Generalization to Weighted Events

Let's now assume that there are two subsamples 1 and 2 with weights w_1 and w_2, from which the efficiency is determined by

$$\epsilon = \frac{w_1\, n_{1+} + w_2\, n_{2+}}{w_1\, n_{1+} + w_1\, n_{1-} + w_2\, n_{2+} + w_2\, n_{2-}}, \tag{2}$$

where n_{i+} is the number of events in subsample i that fulfill the given condition, and n_{i-} is the number of the events that don't.

Again we can assume that n_{1+}, n_{1-}, n_{2+}, and n_{2-} are independently poisson distributed, with errors $\sqrt{n_{i\pm}}$.

Error propagation then yields

$$\delta\epsilon = \sqrt{\left(\frac{\partial\,\epsilon}{\partial\, n_{1+}}\right)^2 n_{1+} + \left(\frac{\partial\,\epsilon}{\partial\, n_{1-}}\right)^2 n_{1-} + \left(\frac{\partial\,\epsilon}{\partial\, n_{2+}}\right)^2 n_{2+} + \left(\frac{\partial\,\epsilon}{\partial\, n_{2-}}\right)^2 n_{2-}}.$$

The partial derivatives are

$$\begin{aligned}
\frac{\partial\,\epsilon}{\partial\, n_{1+}} &= \frac{w_1(w_1\, n_{1-} + w_2\, n_{2-})}{(w_1\, n_{1+} + w_1\, n_{1-} + w_2\, n_{2+} + w_2\, n_{2-})^2} = \frac{w_1 \sum_{-} w_i}{(\sum w_i)^2}, \\
\frac{\partial\,\epsilon}{\partial\, n_{1-}} &= \frac{w_1(w_1\, n_{1+} + w_2\, n_{2+})}{(w_1\, n_{1+} + w_1\, n_{1-} + w_2\, n_{2+} + w_2\, n_{2-})^2} = \frac{w_1 \sum_{+} w_i}{(\sum w_i)^2},
\end{aligned}$$

with the abbrevations

$$\begin{aligned}
\sum_{+} w_i &= w_1\, n_{1+} + w_2\, n_{2+}, \\
\sum_{-} w_i &= w_1\, n_{1-} + w_2\, n_{2-}, \\
\sum w_i &= w_1\, n_{1+} + w_1\, n_{1-} + w_2\, n_{2+} + w_2\, n_{2-} = \sum_{+} w_i + \sum_{-} w_i.
\end{aligned}$$

Therefore one gets

$$\begin{aligned}
\delta\epsilon &= \sqrt{\frac{w_1^2\, n_{1+} \left(\sum_{-} w_i\right)^2 + w_1^2\, n_{1-} \left(\sum_{+} w_i\right)^2 + w_2^2\, n_{2+} \left(\sum_{-} w_i\right)^2 + w_2^2\, n_{2-} \left(\sum_{+} w_i\right)^2}{\left(\sum w_i\right)^4}} \\
&= \sqrt{\frac{(w_1^2\, n_{1+} + w_2^2\, n_{2+}) \left(\sum_{-} w_i\right)^2 + (w_1^2\, n_{1-} + w_2^2\, n_{2-}) \left(\sum_{+} w_i\right)^2}{\left(\sum w_i\right)^4}}
\end{aligned}$$

and finally

$$\delta\epsilon = \frac{\sqrt{\sum_{+} w_i^2 \left(\sum_{-} w_i\right)^2 + \sum_{-} w_i^2 \left(\sum_{+} w_i\right)^2}}{\left(\sum w_i\right)^2} \tag{3}$$

with the notation

$$\begin{aligned} \sum_{+} w_i^2 &= w_1^2\, n_{1+} + w_2^2\, n_{2+}, \\ \sum_{-} w_i^2 &= w_1^2\, n_{1-} + w_2^2\, n_{2-}. \end{aligned}$$

It is easlly seen that this formula holds also for more than 2 samples with different weights, and that it still can be applied to the case where each event has a different weight, in which case the symbols take the following meanings:

$\sum_{+} w_i$	sum of weights for events that fulfill the condition,
$\sum_{+} w_i^2$	sum of squared weights for events that fulfill the condition,
$\sum_{-} w_i$	sum of weights for events that do not fulfill the condition,
$\sum_{-} w_i^2$	sum of squared weights for events that do not fulfill the condition.
$\sum w_i$	sum of weights for all events,

and equation (2) becomes

$$\epsilon = \frac{\sum_{+} w_i}{\sum w_i}.$$

3 Properties

It is easy to see from equation (3) that scaling up all events by a common factor does not change $\delta\epsilon$, as it should not.

In the case where all events have the same value $w_i = w$, one has in the notation of section 1

$$\begin{aligned} \sum_{+} w_i &= w\, n_+, \qquad & \sum_{+} w_i^2 = w^2\, n_+, \\ \sum_{-} w_i &= w\, n_-, \qquad & \sum_{-} w_i^2 = w^2\, n_-, \\ \sum w_i &= w\, n, \end{aligned}$$

and therefore

$$\delta\epsilon = \sqrt{\frac{(w\, n_+)^2 w^2\, n_- + (w\, n_-)^2 w^2\, n_+}{(w\, n)^4}} = \sqrt{\frac{n_+ n_- (n_+ + n_-)}{n^4}} = \sqrt{\frac{n_+ n_-}{n^3}},$$

so equation (1) is retained.

4 Examples

Let's assume we have a sample of $n = 100$ events from which an efficiency $\epsilon = 0.5$ was determined, i. e. $n_+ = n_- = 50$. From formula (1) we get

$$\delta\epsilon = 0.0500.$$

If we take now 100 events of weight $w_1 = 1$ and an additional 200 events with weight $w_2 = 0.5$, and $n_{1+} = n_{1-} = 50$ and $n_{2+} = n_{2-} = 100$, formula (3) yields

$$\delta\epsilon = 0.0306.$$

This can be compared to the case where we have 200 unweighted events, which gives

$$\delta\epsilon = 0.0354,$$

in other words, 300 events where 200 have a weight of 0.5 are not much better than 200 unweighted events. One can turn equation (1) around, to get an effective number n_{eff} of events that would yield the same $\delta\epsilon$ as the weighted sample of events one has:

$$n_{\mathrm{eff}} = \frac{\epsilon\,(1-\epsilon)}{(\delta\epsilon)^2}. \tag{4}$$

For the case of 100 events of weight 1 and 200 of weight 0.5 this yields

$$n_{\mathrm{eff}} = 267,$$

i. e. the 200 events of weight 0.5 do the same for the statistical accuracy as 167 events of weight 1.

If we now add another 500 events of weight 0.2 to the sample, we arrive at

$$\delta\epsilon = 0.0217,$$

and at

$$n_{\mathrm{eff}} = 530,$$

so these additional 500 events of weight 0.2 are equivalent to 233 unweighted events. The statistical error for 800 unweighted events at $\epsilon = 0.5$ would be

$$\delta\epsilon = 0.0177.$$

5 Error on the Mean Weight

Consider the case where a sample of n unweighted events gets assigned weights w_{i}. What is the error on the mean weight

$$\langle w\rangle = \frac{1}{n}\sum_i w_{\mathrm{i}}?$$

We can simply take

$$(\delta w)^2 = \frac{\langle w^2 \rangle - \langle w \rangle^2}{n} \tag{5}$$

$$= \frac{n \sum_i w_\mathrm{i}^2 - \left(\sum_i w_\mathrm{i} \right)^2}{n^3}. \tag{6}$$

In the special case where all weights are either 0 or 1 (i. e., the events are either accepted or rejected), we have

$$\sum_i w_\mathrm{i} = \sum_i w_\mathrm{i}^2 = n_\mathrm{acc}$$

and regain the familiar expression

$$(\delta w)^2 = \frac{n_\mathrm{acc} \cdot (n - n_\mathrm{acc})}{n^3} = \frac{\langle w \rangle \cdot (1 - \langle w \rangle)}{n}.$$

This formula can for instance be applied to check whether reweighting of events compensates event losses: Consider the case where events are reweighted to remove a trigger bias. In a Monte Carlo simulation, we can compare the complete (unweighted) event sample to a sample, where some events were removed due to a missing trigger (i.e., their weight is 0), and the others have been weighted to compensate that. Therefore, if the method works, the mean weight should be 1.

6 Exercise

6.1 Exercise 1

Prove that for a mixture of weighted event the error on $W = \sum w_i$ is given by

$$\delta W = \sqrt{\sum w_i^2}.$$

6.2 Exercise 2

For two subsamples of weighted events, one can also get two efficiencies from

$$\epsilon_i = \frac{n_{i+}}{n_i},$$

with errors

$$\delta \epsilon_i = \sqrt{\frac{n_{i+}\, n_{i-}}{n_i^3}},$$

which one can combine to get the same ϵ as in equation (2):

$$\epsilon = \frac{w_1\, n_1}{w_1\, n_1 + w_2\, n_2} \epsilon_1 + \frac{w_2\, n_2}{w_1\, n_1 + w_2\, n_2} \epsilon_2.$$

Calculate the error of ϵ for this method and compare the two results.

Note that this approach cannot be generalized to the case where all events have different weights.