Lie Algebras and Representations in Theoretical Physics

1) Invitation: Everything about SU(2)
2) A tour of SU(3)
3) Lie Algebras
4) Representations
5) Applications to String theory

Invitation : Everything about SU(2)

Commutation relations $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$

Define $H = \sigma_3$; $E^{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$

$$[H, E^{\pm}] = \pm 2E^{\pm} \qquad [E^+, E^-] = H$$

This basis is particularly useful

From QM, we know that the object they naturally act on are spin states

$$V_1 = \{ |\uparrow\rangle, |\downarrow\rangle \}$$

$$E^+ |\uparrow\rangle = 0 \qquad E^+ |\downarrow\rangle \propto |\uparrow\rangle$$

$$E^- |\uparrow\rangle \propto |\downarrow\rangle \qquad E^- |\downarrow\rangle = 0$$

$$H |\uparrow\rangle = + |\uparrow\rangle \qquad H |\downarrow\rangle = - |\downarrow\rangle$$

All other states obtained by tensor product

$$V_n = V_1^{\otimes n} = \mathrm{span}(|\uparrow\uparrow\ldots\rangle, |\downarrow\uparrow\ldots\rangle)$$

We know this is not the end of the story

$$V_2 = \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$$

Organised in representations of $SU(2)$

$$h = \sum_{i=1}^{n} H_i \qquad H_i = \mathbb{1} \otimes \mathbb{1} \otimes \ldots H \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$$

acts on i-th side

$$e^{\pm} = \sum_{i=1}^{n} E_i^{\pm}$$

We know

$e^{+}|\uparrow\uparrow\rangle = 0$ Heighest weight

$$H|\uparrow\uparrow\rangle = (H_1 + H_2)|\uparrow\uparrow\rangle = 2|\uparrow\uparrow\rangle$$

$$e^{-}|\uparrow\uparrow\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle =$$

$$e^{-}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = |\downarrow\downarrow\rangle \qquad e^{-}|\downarrow\downarrow\rangle = 0$$

decompose into irreps

$$V_n = \{ |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle \} \oplus \{ |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \}$$

Label states by h-eigenvalue

$$h|w\rangle = w|w\rangle$$

"math" convention

better than physics

in physics notation $w = 2(\text{spin})$

$|w\rangle = |m, s\rangle$

$e^{\pm}$ decrease or increase spin

Eigenvalues $h(e^{\pm}|w\rangle) = [h, e^{\pm}]|w\rangle + e^{\pm}w|w\rangle$

$= (w \pm 2)e^{\pm}|w\rangle$

$\rightarrow |w \pm 2\rangle = e^{\pm}|w\rangle$

physics notation $e^{\pm}|m, s\rangle = |m, s \pm \frac{1}{2}\rangle$

Representation ("singlet", "doublet", "triplet") defined by <u>Highest weight state</u> (HWS)

$$E^+|w\rangle = 0 \qquad \left(E^+|\uparrow\rangle = 0 \right)$$

Multiplet obtained by e^-

$$0 \xleftarrow{e^+} |w\rangle \underset{e^+}{\overset{e^-}{\rightleftarrows}} |w-2\rangle \underset{e^+}{\overset{e^-}{\rightleftarrows}} |w-4\rangle \rightarrow \cdots \rightarrow |-w\rangle \rightarrow 0$$

<u>lowest weight</u> : $e^-|w\rangle = 0$

Note that the HWS must exist

if the representation is finite

$V = \text{span}\{|p\rangle, |-+1\rangle, \dots |q\rangle\}$

then take $|a\rangle \in V$ then $(E^+)^n |a\rangle$ is lin indep of $(E^+)^{k-1}|a\rangle$. Since $\dim V < \infty$, $\exists\, k, l$ such that $(E^+)^k |a\rangle = 0 = (E^-)^l |a\rangle$.

Casimir element $C = \frac{1}{2} H + \{E_+, E_-\}$

$[C, X] = 0 \;\; \forall \; X \in su(2)$

on highest / lowest state $|q\rangle, |p\rangle$

$C|q\rangle = \frac{1}{2}(H^2 + 2H + 4E_-E_+)|q\rangle = \frac{1}{2} q(q+2)$

$C|p\rangle = \frac{1}{2}(H^2 - 2H + 4E_+E_-)|p\rangle = \frac{1}{2} p(p-2)$

but $C|p\rangle = C(E_-^{p+q})|q\rangle = E_-^{p+q} C|q\rangle$

$\Rightarrow q(q+2) = p(p-2) \Leftrightarrow (p+q)\underbrace{(p-q-2)}_{q>p} = 0$

$\Rightarrow p = -q \Rightarrow$ an $su(2)$ rep always has $2p+1$ states if $U = \begin{pmatrix} |w_1\rangle \\ |w_2\rangle \\ \vdots \\ \vdots \end{pmatrix} \subset V$

form a d-dim rep of $SU(2)$

$$V_i \xrightarrow{SU(2)} V_i' = T_I^{\;J} V_J$$

$$T_I^{\;J} = \langle w_J | \alpha h + \beta e^+ + \gamma e^- | w_i \rangle$$

upshot: triplet $\{E^-, H, E^+\}$ and $\exists w \; s.t. \; e^+|w\rangle = 0$

→ we know everything about any rep of $SU(2)$:

* All states $|w_i\rangle \propto (e^-)^k |w\rangle$
* $T_I^{\;J}$

We don't really care about T_I^J. All we need to know is that

$T : V \longrightarrow V$ (it just reshuffles states doesn't create new ones)

Math: T is an endomorphism

preserves structure $([\cdot,\cdot])$ and remains inside V.

<u>Goal</u>: do the same for <u>any</u> algebra $\mathfrak{g}$

1) Pick algebra
2) find basis that contains $\{E_i^+, H_i, E_i^-\}$

$\mathfrak{g} \supset su(2) \oplus su(2) \oplus su(2) \ldots$

3) find structure for other generators

↳ captured in "root system"

→ all data in "Dynkin diagram" 

4) representations $E_i^+ |\mu_1, \ldots, \mu_r\rangle = 0$

rest obtained via ladder E_i^-

2) A tour of SU(3)

$$SU(3) = \{ X \in GL(3) \mid X = X^{\dagger},\ \mathrm{tr}\, X = 0 \}$$

$$\dim(SU(3)) = 3^2 - 1 = 8$$

Basis: Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Notice $\exists$ Pauli matrices! Define

$$E_1^{\pm} = \frac{1}{2}(\lambda_1 \pm i\lambda_2) \qquad H_1 = \lambda_3$$

$$E_2^{\pm} = \frac{1}{2}(\lambda_6 \pm i\lambda_7) \qquad H_2 = \frac{1}{2}(-\lambda_3 + \sqrt{3}\lambda_8)$$

$$V^{\pm} = \frac{1}{2}(\lambda_4 \pm i\lambda_5)$$

The commutation relations are then

$$[H_1, H_2] = 0$$

$$\left.\begin{aligned} &[H_i, E_i^{\pm}] = 2E_i^{\pm} \\ &[E_i^+, E_i^-] = H_i \end{aligned}\right\} 2\ SU(2)\ !$$

not decoupled

Cartan matrix

$$[H_i, E_j^{\pm}] = \pm A_{ji} E^{\pm} \qquad A_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

rest behaves nicely

$$[E_1^{\pm}, E_2^{\pm}] = V^{\pm}$$

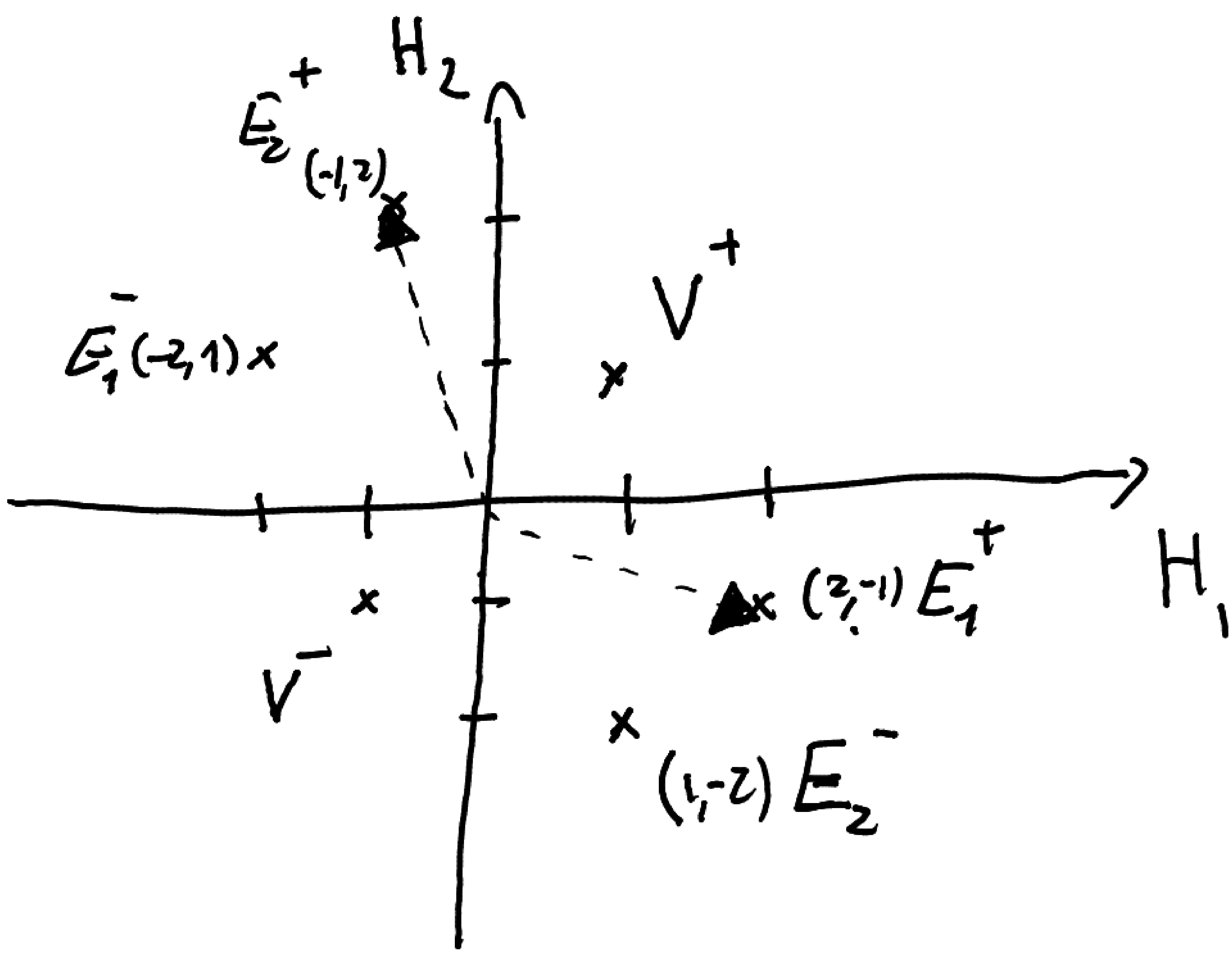
$$[E_1^{\pm}, E_2^{\mp}] = 0$$

$$[V^+, V^-] = H_1 + H_2$$

$$[H_i, V^{\pm}] = \pm V^{\pm}$$

2.2 Arranging Eigenvalues

Let us plot H_i-eigenvalues



We transformed commutators into eigenvalues

$$E_1^+ = E_{\alpha_i} \qquad \alpha_i = (2,-1)$$

$$E_2^+ = E_{\alpha_i} \qquad \alpha_i = (-1,2)$$

$$E_i^- = E_{-\alpha_i}$$

$$[E_\alpha, E_\beta] \sim E_{\alpha+\beta}$$

$$V^\pm = E_{\pm(\alpha^1+\alpha^2)}$$

if $\alpha = 0$ $\quad E_\alpha = a H_1 + b H_2$

Even though we started with $SU(3)$, we translated things into multiple $SU(2)$ problems

representations will look like

$$|w_1, w_2\rangle$$

$$\text{HWS} \quad E_i^+ |w_1, w_2\rangle = 0$$

$$E_1^- \swarrow \qquad \searrow E_2^-$$

....

Let us formalise this for <u>any</u> algebra.

3) Lie Algebras

def: $\mathfrak{g} = \text{span}(\{X_A\})$ with $[X_A, X_B] = i f_{ABC} X_C$

$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bilinear + Jacobi

In our case, just matrix commutator

$A = 1, \dots, \dim(\mathfrak{g})$

Related definitions

* $\mathfrak{g}' \subset \mathfrak{g}$ a <u>subalgebra</u> if $[X, Y] \in \mathfrak{g}'$ $\forall X, Y \in \mathfrak{g}'$
* $I \subset \mathfrak{g}$ an <u>ideal</u> if $\forall X \in \mathfrak{g}$ and Y in $Y \in I$ then $[X, Y] \in I$

  (cannot escape I by commutation)
* $\mathfrak{g}$ is simple if it has no non-trivial ideal

  (no subalgebra is more interesting)

* semi-simple if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$

→ Simple algebras are the "atoms" of all algebras

simple: $SU(2)$

semi-simple: $\mathfrak{g}_{SM} = SU(3) \oplus SU(2) \oplus U(1)$

not semi simple: Poincaré $\left([M^{\mu}_{\nu}, P^{\rho}] \sim P^{\rho} \right)$

if we know everything about translations AND Lorentz, we know everything about Poincaré

Last nomenclature. Recall the adjoint action

$\forall \; X \in \mathfrak{g}$

$$ad_X : \mathfrak{g} \to \mathfrak{g}$$
$$Y \to [X, Y]$$

3.1: Structure of Lie Algebras

Cartan subalgebra: maximal commuting subalgebra $\mathfrak{h} \subset \mathfrak{g}$

$\mathfrak{h} = \mathrm{span}(H_i)$ with $[H_i, H_j] = 0$ $\quad H^+ = H$

rank : $r = \mathrm{rk}(\mathfrak{g}) = \dim(\mathfrak{h})$ $\qquad i = 1, \dots, \mathrm{rk}(\mathfrak{g})$

write $\mathfrak{g} = \mathrm{span}(\{H_i\}, \{E_\alpha\})$

commuting matrices, can digonalise. Via ad map

As vector space $\mathrm{ad}_H X = k X$; $k \in \mathbb{R}$

We write $[H_i, E_\alpha] = a_i \, E_\alpha$ $\quad \forall \, H \in \mathfrak{h}$

α is called a root (vector), with comp

$$\alpha = \begin{pmatrix} a_i \\ \vdots \\ a_r \end{pmatrix}$$

Each non-Cartan generator is defined by a root vector

Jacobi

$$[H_i,[E_\alpha,E_\beta]] = [E_{\alpha_{(i)}},[H,E_\beta]] - [E_\beta,[H,E_\alpha]]$$

$$= (a_i + b_i)[E_\alpha,E_\beta]$$

Two possibilities

$\alpha+\beta=0$: $[E_\alpha,E_{-\alpha}]\in h$

$a_i+b_i\neq 0\ \forall i$ $\exists\ \gamma=\alpha+\beta$ st $[E_\alpha,E_\beta]=N_{\alpha+\beta}E_{\alpha+\beta}$

NB: # different α = $\dim(g)-rk(g)$

many $N_{\alpha+\beta}=0$

for SU(3) we had

$$\alpha\in\left\{\alpha_1=\begin{pmatrix}2\\-1\end{pmatrix},\ \alpha_2=\begin{pmatrix}-1\\2\end{pmatrix},\ -\alpha_1, -\alpha_2, \pm(\alpha_1+\alpha_2)\right\}$$

so $N_{2\alpha_1}=0=N_{\alpha_1+2\alpha_2}$ etc...

3.3 The killing form

From QFT, recall killing form

$$K(X,Y) = \mathrm{Tr}\,(\mathrm{ad}_X\, \mathrm{ad}_Y)$$

$$K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$
$$(X,Y) \to K(X,Y)$$

if $X = m^A X_A$; $Y = n^A X_A$

$K(X,Y) = m^A g_{AB} n^B$ $\qquad g_{AB} \propto f_A{}^{CD} f_{BCD}$

properties: symmetric $K(X,Y) = K(X,Y)$

bilinear $K(aX+Y,Z) = a\,K(X,Z) + K(Y,Z)$

non-degenerate $K(X,Y) = 0 \;\forall Y$

$\Rightarrow X = 0$

In our basis, can show

only non-vanishing comp g_{ij}, $g_{\alpha,-\alpha}$

It further satisfies $K([X,Y],Z) = K(X,[Y,Z])$

Assume $[E_\alpha, E_{-\alpha}] = H_\alpha = \lambda_\alpha^i H_i$

normalization

$$K(H,H_i) = \lambda_\alpha^j g_{ij}$$

$$\begin{aligned} K(H_\alpha,H_i) &= K([E_{+\alpha}, E_{-\alpha}], H_i) \\ &= K(E_{-\alpha}, [E_\alpha, H_i]) \\ &= a_i \, K(E_{-\alpha}, E_\alpha) = a_i \, \overbrace{K(E_{-\alpha}, E_\alpha)}^{=1} \end{aligned}$$

$$a_i = g_{ij}\lambda_\alpha^j \qquad H_\alpha = a_i g^{ij} H_j$$

$g^{ij} = (g^{-1})_{ij}$ (always possible since K non degen)

Why is this useful?

$$\begin{aligned} [H_\beta, E_\alpha] &= \lambda_\beta^i [H_i, E_\alpha] = \lambda_\beta^i a_i E_\alpha \\ &= (b_i g^{ij} a_j) E_\alpha \end{aligned}$$

looks like a scalar product!

Denote $\Phi = \{\alpha \mid \exists\, E_\alpha\}$

$$\langle\alpha,\beta\rangle : \Phi \times \Phi \longrightarrow \mathbb{C}$$
$$(\alpha,\beta) \longrightarrow \langle\alpha,\beta\rangle = a_i g^{ij} b_j$$

We proved the existence of Cartan-Weyl basis

$$[E_\alpha, E_{-\alpha}] = H_\alpha$$

$$[H_\alpha, E_\beta] = \langle\alpha,\beta\rangle E_\beta$$

$$[E_\alpha, E_\beta] = N_{\alpha+\beta} E_{\alpha+\beta} \quad \text{if } \alpha+\beta \neq 0$$

$$[H_\alpha, H_\beta] = 0$$

we can do slightly better

Recap 1st Lecture

$$\mathfrak{g} \rightsquigarrow \mathfrak{h} \rightsquigarrow \{H_i, E_\alpha\}$$

Cartan-Weyl basis

$\forall \alpha, \exists \quad H_\alpha = [E_\alpha, E_{-\alpha}] = a^i H_i$

$$[H_\alpha, E_\beta] = \langle \beta, \alpha \rangle E_\beta$$

$$[E_\alpha, E_\beta] = N_{\alpha+\beta} E_\beta \quad \text{if } \alpha+\beta \neq 0$$

talk about roots rather than $\mathfrak{g}$ itself

(@ Thomas: cheated a bit, Killing allowed us to write

$$[H_\alpha, E_\beta] = \langle \beta, \alpha \rangle E_\beta$$

dot product for roots

Pick α^i such that

$$[E_i^+, E_j^-] = H_i \qquad E_{\alpha^i}^{\pm} = E_i$$

$$[H_i, E_i^{\pm}] = \langle \alpha_i, \alpha_j \rangle E_i^{\pm}$$

α_i : simple roots $\quad i = 1, \dots, \mathrm{rk}(\mathfrak{g})$

can then define coroot

$$\alpha^{\vee} = \frac{2}{\langle \alpha, \alpha \rangle} \alpha$$

rescale $H_i \rightsquigarrow \dfrac{2 H_i}{\langle \alpha_i, \alpha_i^{\vee} \rangle}$

We obtain the Serre-Chevalley

$$[E_i^+, E_i^-] = H_i \qquad i=1,\dots,\mathrm{rk}(\mathfrak{g})$$

$$[H_i, E_j^\pm] = \pm \langle \alpha_j, \alpha_i^\vee \rangle E_j^\pm$$

$$[E_\alpha, E_\beta] = N_{\alpha+\beta} E_{\alpha+\beta} \text{ if } \alpha+\beta \neq 0$$

if $i=j$ $\quad [H_i, E_i^\pm] = \pm \frac{2\langle \alpha_i, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \pm 2 E_j^\pm$

$\Rightarrow$ We found $\mathrm{rk}(\mathfrak{g})$ $SU(2)$!

We achieved something impressive

- can find $SU(2)^r$ in any $\mathfrak{g}$
- Everything encoded into <u>Cartan matrix</u> $\quad A_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$

classify $\mathfrak{g}$ $\longleftrightarrow$ classify A_{ij}

complicated $\qquad$ (easy !)

4) Root Systems

root system : if $\mathfrak{g} = \text{span}(H_i, E_\alpha)$, $\Phi = \{\alpha\}$
Φ is the root system

Since $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$

any root $\alpha \in \Phi$ is $\alpha = n^i \alpha_i$ $n^i \in \mathbb{Z}$

$\Pi = \{\alpha_1, \dots, \alpha_r\}$ are <u>simple roots</u>

root systems are very constrained

<u>root-string lemma</u>: Consider $\alpha, \beta \in \Phi$

1) $\exists$ $p, q \in \mathbb{N}$ st

$\beta + k\alpha \in \Phi$ $\forall$ $k = -q, \dots, p$

($\exists$ a root string of $\cancel{\alpha}$ passing through β)

2) $p - q = \langle \beta, \alpha^\vee \rangle$

proof: choose $H_\alpha = [E_\alpha, E_{-\alpha}]$; $[H_\alpha, E_\beta] = \langle \beta, \alpha^\vee \rangle E_\beta$

1) Since $[E_\alpha, E_\beta] = N_{\alpha+\beta} E_{\alpha+\beta}$ for $\alpha+\beta \neq 0$

we can nest commutators until

$$\underbrace{[E_\alpha [\ldots, E_\alpha}_{q+1}, E_\beta] = 0$$

and similarly for $(p+1)$ nested $E_{-\alpha}$

2) H_α $E_{\pm\alpha}$ form $SU(2)$

$E_{\beta+q\alpha}$ HWS of representation

$$0 = e^+ |w\rangle \leftrightarrow \mathrm{ad}_{E_\alpha} E_{\beta+q\alpha} = [E_\alpha, E_{\beta+q\alpha}] = 0$$

weight $[H_\alpha, E_{\beta+q\alpha}] = (\langle \beta, \alpha^\vee \rangle + q\langle \alpha, \alpha^\vee \rangle) E_{\beta+q\alpha}$
$= (\langle \beta, \alpha^\vee \rangle + 2q) E_{\beta+q\alpha}$

lowest weight $[H_\alpha, E_{\beta-p\alpha}] = (\langle \beta, \alpha^\vee \rangle - 2p) E_{\beta-p\alpha}$

From before, we know if HWS $|w\rangle$ then LWS $|-w\rangle$

$$\langle \beta, \alpha^\vee \rangle - 2p = -(\langle \beta, \alpha^\vee \rangle + 2q)$$

$$\Rightarrow \langle \beta, \alpha^\vee \rangle = p - q$$

Consequences:

1) $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z} \quad \forall\, \alpha, \beta \in \Phi$!!!

$\hookrightarrow A_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle \in \mathrm{Mat}(r, r, \mathbb{Z})$

2) $\langle \alpha, \alpha^\vee \rangle = 2 \Rightarrow p - q = 2$

$\beta = k\alpha \Rightarrow \beta = \pm \alpha$ (see next page)

3) $\langle \beta, \alpha^\vee \rangle = p - q \in \{0, \pm 1, \pm 2, \pm 3\}$

$\hookrightarrow A_{ii} = 2 \quad A_{ij} \in \{0, -1, -2, -3\}$ (see proof in Cahn)

4) Can prove: for root string $\beta + k\alpha$ with

$\max(k) = q$

$N_{\alpha+\beta} = \pm(q+1)$ (proof in Cahn) prop: if $\alpha, \beta \in \Phi$ and $\beta = m\alpha$, then $m = \pm 1$.

* We know that $\alpha, -\alpha \in \Phi$ by assumption. We only need to exclude $\beta = m\alpha$ for $|m| > 1$

* $\exists$ $p, q > 0$ st $\langle \beta, \alpha^\vee \rangle = m \langle \alpha, \alpha^\vee \rangle$
$= 2m \overset{!}{=} p - q$

and $\beta + k\alpha \in \Phi$ $\forall$ $k = -p, \dots, q$

we have $p = 2m + q$

$$\beta - p\alpha = (m - (q + 2m))\alpha$$
$$= (-m - q)\alpha$$

$$\beta + q\alpha = (m + q)\alpha$$

if $|m| > 1$ then the root string has

$-(m+q) \leq k \leq q$ $\Rightarrow$ passes through 0

but $0 \notin \Phi$. The root lemma say $\beta + k\alpha \in \Phi$ $\forall -p \leq k \leq q$
$\rightarrow$ contradiction, we cannot have $|m| > 1$
as we would have $\beta + k\alpha = (m - m)\alpha \notin \Phi$

4.1 Classification of finite root systems

if we know the Cartan matrix

$$A_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$$

we know everyting. All roots reconstructed via root strings

We already know a lot

$$A_{ii} = 2 \qquad A_{ij} \in \{0, -1, -2, -3\}$$

Let us construct all of them

$\langle \cdot, \cdot \rangle$ is a scalar product

$$\langle \alpha_i, \alpha_j \rangle^2 \leq \langle \alpha_i, \alpha_i \rangle \langle \alpha_j, \alpha_j \rangle \quad \text{(Schwarz inequality)}$$

equality only if $\alpha_i \propto \alpha_j$, but can't happen. Dividing by the norms

$$A_{ij} A_{ji} < 4 \quad i \neq j \quad \text{(no sum)}$$

$\Rightarrow$ if $A_{ij} = -3, -2, -1 \Rightarrow A_{ji} = -1 \quad i \neq j$

From there, we only need to recall $\langle , \rangle$ descends from $K(,)$. The Killing form is positive definite, this implies

$A_{ij} > 0$ (positive definite matrix)

We have

1) $A_{ij} > 0$
2) $A_{ii} = 2$; $A_{ij} = -3, -2, -1, 0$
3) $A_{ij} < -1 \Rightarrow A_{ji} = -1$

For $SU(3)$ we found $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

part of a long series

$$A_{ij} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \qquad \begin{matrix} A_n \text{ series} \\ SU(n+1) \end{matrix}$$

Dynkin diagrams

Given a Cartan matrix A_{ij}, we can represent it graphically

1) a node ○ for $A_{ii}=2$ (so $\mathrm{rk}(g)$ nodes)
2) if $A_{ij}\neq 0$ the nodes are connected by $A_{ij}A_{ji}$ edges
3) if $A_{ij}A_{ji}>1$, and $\langle\alpha_i,\alpha_i\rangle < \langle\alpha_j,\alpha_j\rangle$ add an arrow point from i to j

example

$SU(3)$: $A_{ij}=\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$SU(n)$: $A_{ij}=\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$

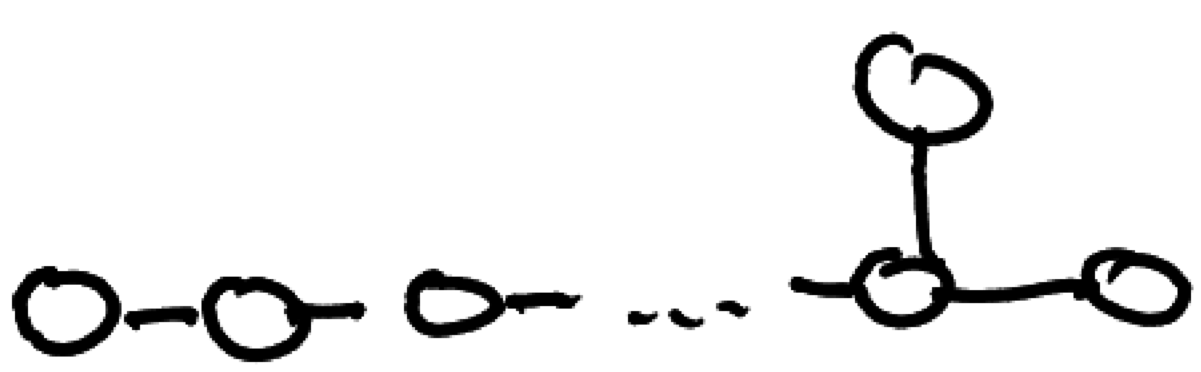
All possibilities

Classical

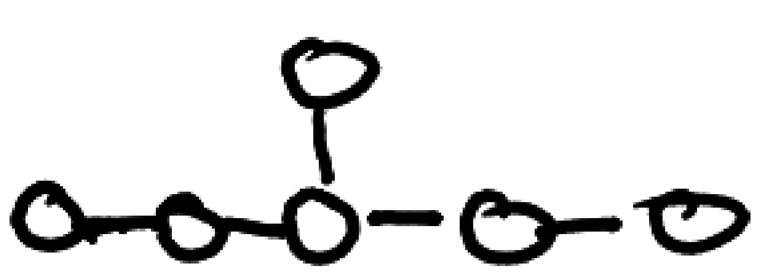
A_n :  $SU(n)$ $\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & -1 & 2 & -2 \\ & & & -1 & 2 \end{pmatrix}$

B_n :  $SO(2n+1)$

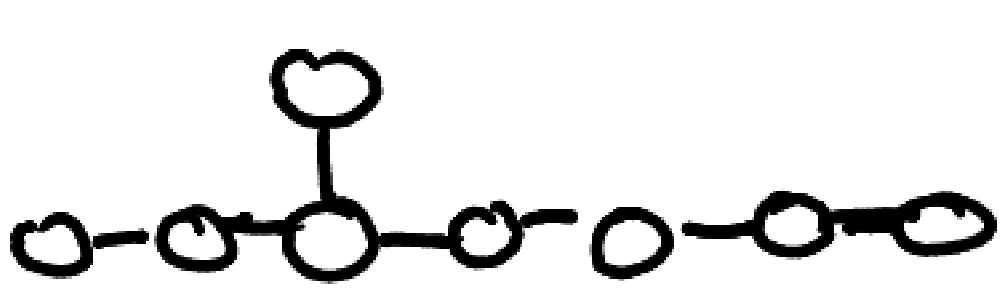
C_n :  $SP(n)$ $\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & \ddots & -1 \\ & & -2 & 2 \end{pmatrix}$

D_n  $SO(2n)$ $\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & 2 & -1 & -1 \\ & & -1 & 2 & 0 \\ & & -1 & 0 & 2 \end{pmatrix}$

Exceptional

E_6  E_6

E_7  E_7

E_8  E_8

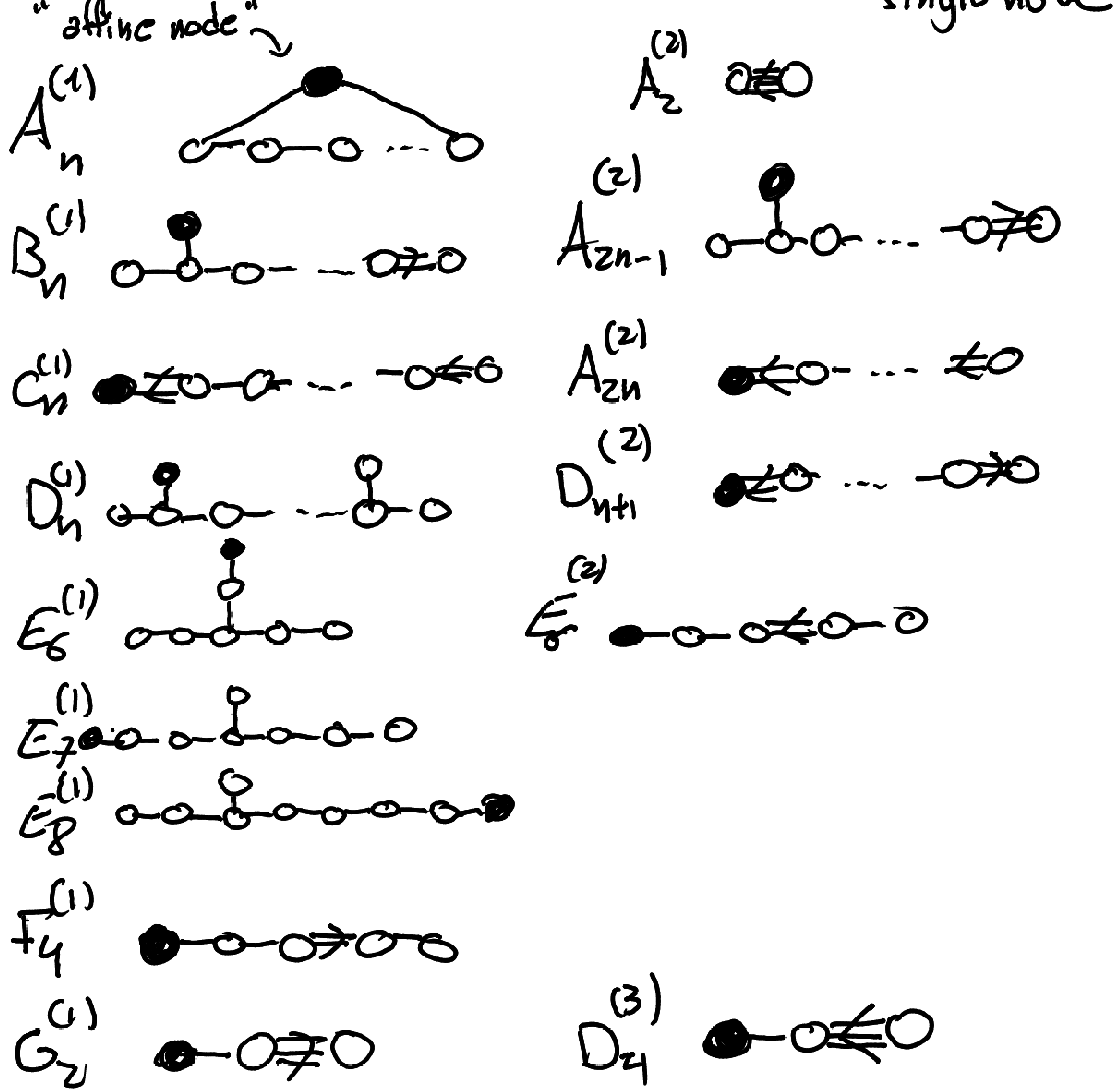
F_4  F_4 $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$

G_2  G_2 $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

Could redo the whole thing, but relaxing positive definiteness to semi-positive definite

$\to$ (twisted) affine algebras $\tilde{\mathfrak{g}}$

classification easy (w/o proof) only add single node



Cartan matrix easy, but $\dim(\mathfrak{g}) = \infty$

some roots satisfy $\beta = k\alpha \ \forall k \in \mathbb{N}$ contrary to finite

Important

A_n, D_n, E_n satisfy $A^t_{ij} = A_{ij}$

$\longrightarrow$ simply laced ADE classification

Most beautiful and deepest result in math

* singularities of certain manifolds
  $\rightsquigarrow$ CY $\rightsquigarrow$ profound relation to strings

* Classifications of certain CFTs in $d = 2, 6$

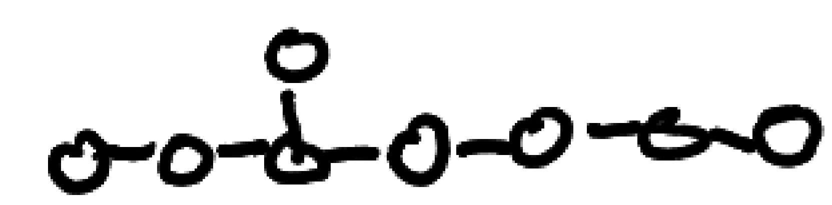
$$\text{OPE}: \ \Phi_a \times \Phi_b = N_{abc} \Phi_c \qquad N^c_{ab} > 0$$

in certain cases $\Phi_a \times \Phi_{\text{ref}} = C_{ab} \Phi_c$

$C_{ab} = C_{ba} \quad C_{ab} > 0 \quad C_{aa} = 2 \quad \Rightarrow$ ADE

5) Representation theory

So far: $\mathfrak{g} \rightsquigarrow \mathfrak{h} \oplus \{E_\alpha\} \rightsquigarrow \Phi$

$\rightsquigarrow A_{ij} \rightsquigarrow$ 

* Everything is encoded into Dynkin diagram
* Even though $\mathfrak{g}$ is itself important, we don't truly care about it as physicists

representations: peeling one more layer of the onion

in physics, we don't really care about the symmetry itself, but how it <u>acts on things</u>

Mathematically: $V \xrightarrow{\mathfrak{g}} V \quad (\phi^a \to T^a{}_b \phi^b)$

Def: A <u>representation</u> is a pair (P, V) with

$$\begin{aligned} P: \mathfrak{g} &\longrightarrow GL(V) \\ X &\longrightarrow P(X) \end{aligned} \quad \text{st} \quad [P(X_A), P(X_B)] = i f_{ABC} P(X_C)$$

(*we work only w/ $\dim(V) < \infty$)

In practice V defines ρ intrinsically

Physicist notation: if $\dim(V)=R$ denote rep as $\underline{R}$ (boldface)

for SU_3 $\underline{3}$ is fund, $\underline{8}$ is adj
Antifun: $\underline{\bar{3}}$ (almost) unique labelling

Given V, we can work directly in Chevalley basis $h_i = \rho(H_i)$ $e_i = \rho(E_i)$

Since h_i is representation of Cartan H_i, $\exists$ a basis of V st any element takes the form

$|\vec{\mu}\rangle = |\mu_1, \dots, \mu_r\rangle \in V$ $\quad h_i|\mu\rangle = \mu_i|\mu\rangle$

the vector $\vec{\mu}$ (we will drop the arrow) is called a <u>weight</u>

we already know a lot about it!

Write $H_\alpha = [E_\alpha, E_{-\alpha}] \leftarrow su_2$!

if $\alpha = n^i \alpha_i$ w/ $n^i \geq 0$ ← positive root

$\exists\, k$ st $E_\alpha^k |p\rangle = 0 \quad \forall\, p$

can basically redo everything we did for root. In particular

$$H_\alpha |p\rangle = \langle p, \alpha^\vee \rangle |p\rangle$$

# of Roots and Weights

We need a good basis.

simple roots

Define the root lattice $Q = \mathrm{span}_{\mathbb{Z}}(\{\alpha_i\})$

coroot lattice $Q^\vee = \mathrm{span}_{\mathbb{Z}}(\{\alpha_i^\vee\})$

$\Phi = \{\alpha\} \subsetneq Q$ (Φ only has dim-th elements)

The <u>dual basis</u> is the set of ω^i st

$$\langle \omega^i, \alpha_j^\vee \rangle = \delta^i_j$$

the ω^i are the <u>fundamental weights</u> and span the weight lattice

$$P = \mathrm{span}_{\mathbb{Z}}(\{w_i\})$$

Why is this useful?

Write $\mu = \mu_i \omega^i$

$$h_i|\mu\rangle = \langle \mu, \alpha_i^\vee \rangle |\mu\rangle = \mu_j \langle \omega^j, \alpha_i^\vee \rangle |\mu\rangle$$
$$= \mu_j \delta_i^j |\mu\rangle = \mu_i |\mu\rangle$$

Similarly to SU(2), we define <u>Highest weight</u> μ if

$$e_i^+|\mu\rangle = 0 \quad \forall i$$

and the lowest-weight state is $|\rho\rangle$ if

$$e_i^-|\rho\rangle = 0 \quad \forall i$$

How to construct rep of $\mathfrak{g}$?

Use Chevalley basis; do like SU(2)

Take any weight $|p\rangle$. What is the weight of $e_i^-|p\rangle$

$$h_j e_i^{\pm}|p\rangle = ([h_j, e_i^-] \pm e_i^- h \;)|p\rangle$$

$$= (\pm A_{ij}\, e_i^- + p_j\, e_i^-)\,|p\rangle$$

$$|p\rangle \xrightarrow{e_i^{\pm}} |p \pm \alpha_i\rangle \qquad \alpha_i = i\text{-th colum of } A_{ij}$$

$e_i^{\pm}$ allow us to reconstruct everything from HWS! Like SU(2)

Important difference:

* we have $\mathrm{rk}(\mathfrak{g})$ ν_i (rather than 1)
* shifting w/ e_i^- also changes ν_j $i \neq j$

Which weights is a highest weight?

<u>def</u>: a weight $\nu = \nu_i \omega^i$ is

1) <u>integral</u> if $\nu \in P = \mathrm{span}_{\mathbb{Z}}(\{\omega^i\})$
($\nu_i \in \mathbb{N}$)

2) <u>dominant</u> if $\langle \nu, \alpha_i^\vee \rangle = \nu_i \geq 0$
and ν integral ($\nu_i \in \mathbb{N}$)

<u>prop</u>: every dominant weight is the highest weight of a representation

Constructing any representation

1) Pick a dominant weight $\mu = \mu_i \omega^i > \mu_i \geq 0$

   $|\mu\rangle$ is a HWS

2) initial step $|\rho\rangle = |\mu\rangle$

3) for all $\rho_i \geq 0$ find all $|\lambda\rangle = |\rho - k\alpha_i\rangle$

   $k = 0, \ldots, \mu_i$

4) Apply step 3 recursively until all new $|\lambda\rangle$ have $\lambda_i < 0$

Example

$SU_3 = A_2$ $\circ\!-\!\circ$ $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

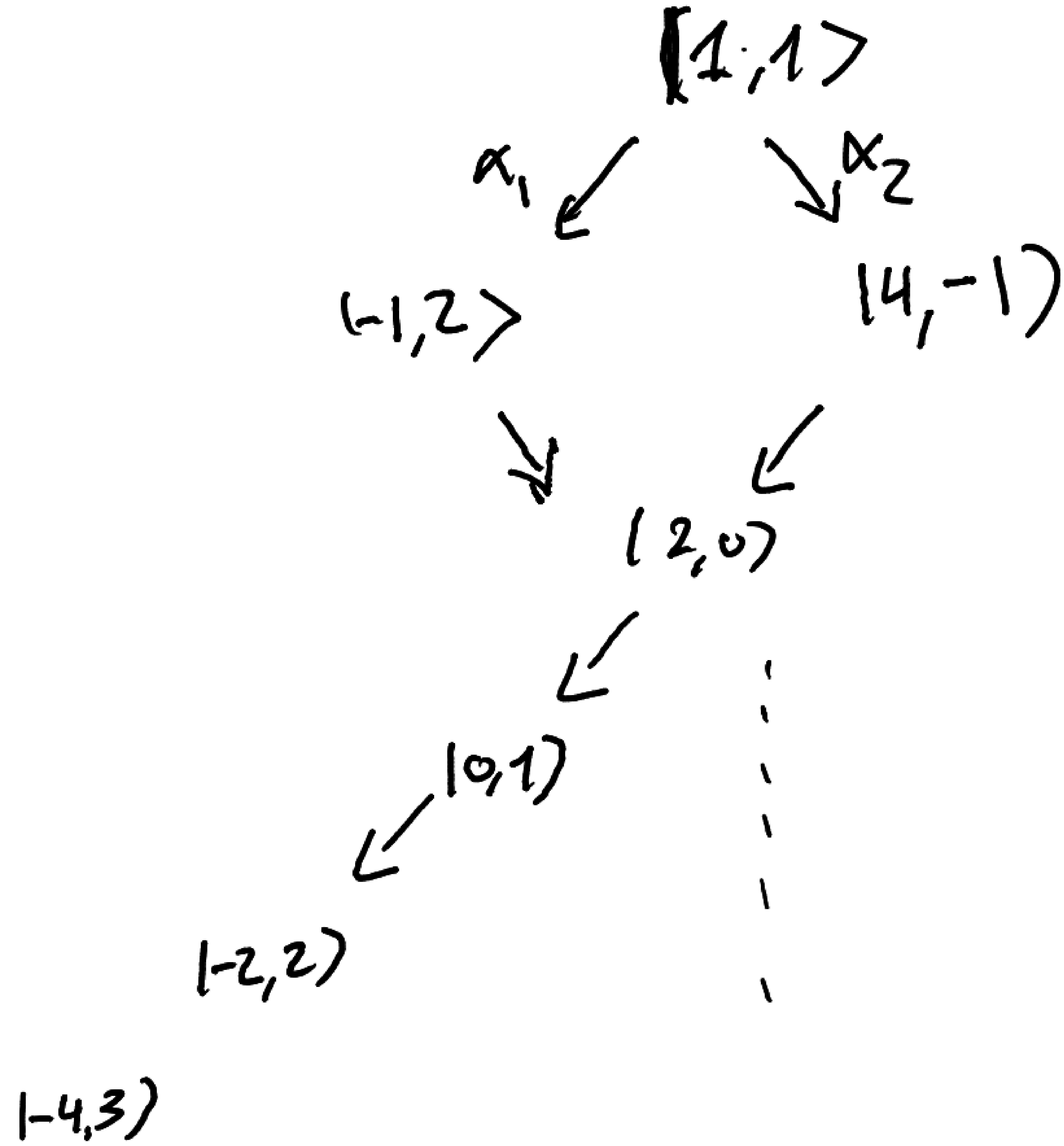
$|\mu\rangle = |1,1\rangle$

$$
\begin{tikzcd}
 & {|1,1\rangle} \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\
{|-1,2\rangle} \arrow[dr, "\alpha_2"'] & & {|2,-1\rangle} \arrow[dl, "\alpha_1"] \\
 & (0,0) \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\
(-2,1) \arrow[dr, "\alpha_1"'] & & (1,-2) \arrow[dl] \\
 & (-1,-1) &
\end{tikzcd}
$$

$\rightarrow$ $|1,1\rangle$ is the HWS of the $\underline{8}$ of SU_3

Example $\mathfrak{g}_2$ $\heartsuit \neq 0$ $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

$|\mu\rangle = |2,0\rangle$



get $\dim(V_{|1,1\rangle}) = 64$

$\rightarrow$ rep depends highly on A_{ij}

Conjugation

Conjugation flips sign of weights

$C : |\mu\rangle \to |-\mu\rangle$ (seen from $(T^C_{ij}) = -T^*_{ij}$)

since HW is dominant, the highest weight changes

example A_2

$$
\begin{tikzcd}
(1,0) \arrow[d, "\alpha_1"'] & & (0\ 1) \arrow[d, "\alpha_2"] \\
(-1,1) \arrow[d, "\alpha_2"] & \xrightarrow{\ C\ } & (1,-1) \arrow[d, "\alpha^1"'] \\
(0,-1) & &
\end{tikzcd}
$$

3 $\qquad\qquad\qquad\qquad$ $\overline{3}$

Can obtain new rep from conjugation

Some reps are real

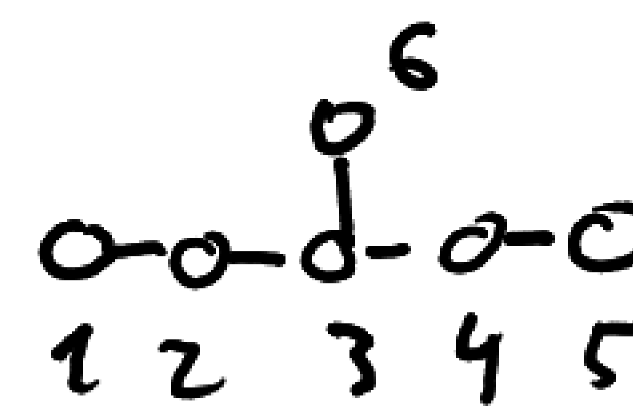
$$R = \overline{R} \Leftrightarrow CV = \{|-\mu\rangle\} = \{|\mu\rangle\} = V$$

depends on algebra

$A_r : \mu_i \to \mu_{n-i+1}$

$D_{r=2n+1} : \mu_r \to \mu_{r-1}$

$E_6 : \mu_1 \to \mu_5 \; ; \; \mu_2 \to \mu_3$



all other are real*

* There is a slight refinement between real and pseudo real

Product Representation

there is always a representation from which all other can be obtained

fundamental representation $|\mu\rangle = |1,\dots,0\rangle$

smallest non-trivial rep

like $SU(2)$ all others obtained via tensor products

$$|\mu_{fun}\rangle^{\otimes n} \quad \longleftrightarrow \quad |\uparrow\rangle$$

usually reducible $\quad V_{|\mu\rangle^{\otimes n}} = \bigoplus_i V_{|\mu_i\rangle}$

ladder operators $\quad h_i = \sum_{a=1}^{n} h_i^{(a)} \quad e_i^{\pm} = \sum_{a=1}^{n} e^{(a)\pm}$

1) new weights $h_i|\mu_{new}\rangle = h_i|\mu_1\rangle \otimes \dots \otimes |\mu_n\rangle$

$$= \sum_{a=1}^{n} h_i^{(a)} |\mu_{new}\rangle$$

2) find all HW $e_i^+ |\mu_{new}\rangle = 0$

Hard to do manually

↳ LieART, Slansky...

simple example

$SU(n)$: $n \otimes \bar{n} = Adj \oplus 1$

dim

Subalgebra (very rough)

Take $\mathfrak{g} \rightsquigarrow \Phi = \{\alpha\}$

if $\exists$ subalgebra $\mathfrak{g}' \subset \mathfrak{g}$, we can find $\Phi' = \{\beta\}$ such that $\exists$ projection $\mathfrak{h}' \subset \mathfrak{h}$ and $P : \Phi \to \Phi'$

if we know $\underbrace{P_i^{\ j}}_{\text{rectangular matrix}} \underbrace{\alpha_j = \beta_j}_{\text{simple roots}}$

we can reconstruct the new root system

$$SU_4 : \begin{array}{c} \begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix} \\ \alpha_1 \ \alpha_2 \ \alpha_3 \end{array} \longrightarrow \begin{array}{c} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\ \beta_1 \ \beta_2 \end{array}$$

Can also do that for representations

$$|\mu_1, \dots \mu_{r_g}\rangle \rightsquigarrow |\beta_1, \dots, \beta_{r_{g'}}\rangle$$

New representation reducible

↳ same procedure as before

↳ LieART, Slansky, ...

Can also be semi-simple

$$SU(5) \longrightarrow SU(3) \oplus SU(2) \oplus U(1) = g_{SM}$$

$$\text{Adj}: 24 \longrightarrow (8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0 \oplus$$
$$\oplus (3,2)_{\frac{5}{6}} \oplus (\bar{3},2)_{-5/6}$$

$(1,1)_0$ singlet under all g_{SM}

each rep has a (super) field

$(1,1)_0 \leftrightarrow \Phi$ if $\langle\Phi\rangle \neq 0$

breaks $SU(5) \to g_{SM}$