II. Institut für Theoretische Physik
Universität Hamburg
Winter term 2022/23

Project II

Due date: 3rd of February 2023

- You can use any means you deem necessary to solve the problems, but your submission has to be written by you. Collaboration is encouraged.
- Write your name and matriculation number at the beginning of each page of the document.
- Solutions can be handed in either English or German.
- Refrain from using numerical values before the end of a given question, and always clearly explain where they come from.
- Upload via Moodle by 2023-02-03, 23:59. **Only files in pdf format will be accepted, no exceptions** (no Word documents, no giant zip file with jpegs in it, . . .).
- If more convenient, it is also possible to hand in a paper version. In that case, the deadline is at the beginning of the lecture on 2023-01-31, 16:15. *Note that this means you would have to submit earlier than via Moodle!*

Problem 2.1 (Four-vectors and Einstein's Summation Convention):
Consider a covariant four-vector $a^\mu = (a^0, \vec{a}) = (a^0, a^1, a^2, a^3) \in \mathbb{R}^{1,3}$.

a) Explain the difference between a^μ and a_μ and how they transform under Lorentz transformations.

b) Give the components of a_μ in terms those of a^μ given above.

c) If b^μ is another 4-vector, explain why $a^\mu b_\mu$ will be a Lorentz scalar, and write its expression in terms of the component of a^μ, b^μ.

d) Given spacetime coordinates x^μ, we can define the derivative 4-vector $\partial_\mu = \frac{\partial}{\partial x^\mu}$. It is possible to show that ∂_μ transforms under Lorentz transformations as a contravariant four-vector (you do not need to prove it). Write $\partial_\nu x^\mu$ in components, and show that $\partial_\mu x^\mu = 4$ in all reference frames.

e) We now focus on the four-momentum p^μ.

1) What is the physical meaning of the component p^0?
2) Which two conservation principles does Special Relativity unify compared to Newtonian Mechanics?
3) The relativistic energy $\mathcal{E}$ and relativistic 3-momentum $\vec{p}$ of a particle of mass m moving at a velocity $\vec{v}$ are given by

$$\mathcal{E} = m\gamma(v)c^2 \,, \qquad \vec{p} = m\gamma(v)\vec{v} \,, \tag{1}$$

show that $p^\mu p_\mu = (mc)^2$.

4) Is Einstein's famous formula, $\mathcal{E} = mc^2$ valid in all reference frames? Why?
5) If two particles of mass m and M have 4-momenta p^μ and k^μ respectively, simplify the following expression:

$$(p^\mu + k^\mu)(p_\mu + k_\mu) \,. \tag{2}$$

6) If a particle has a 4-momentum p^μ in a given reference frame $\mathcal{R}$, give the expression of its 4-momentum $(p')^\mu$ in an inertial reference frame $\mathcal{R}'$ going at velocity v along the x-axis with respect to $\mathcal{R}$ (give it for all four spacetime directions!).
7) We have seen in the lecture that the 4-momentum p^μ of massive particles is fundamentally different from that of massless particles. Shortly summarise the difference, and give the expression of p^μ in a convenient frame in both cases.
8) What goes wrong if we try to go to the rest frame of a massless particle?

Problem 2.2 (Charmonium):

In this exercise, use only numerical values as the very last step of your answer!

In 1974 two experiments in the USA using electron-positron colliders independently detected two new particles, the so-called Ψ and Ψ' mesons. This discovery marked an important milestone in our understanding of the Universe as it confirmed the quark model. Indeed, particles like protons and neutrons are not indivisible, but constituted of more elementary particles, the quarks.

In the case of the Ψ and Ψ', a charm quark is bound together with an anti-charm quark by the strong nuclear force. The binding energies of the two mesons are distinct, and they have therefore a different masses:

$$\begin{aligned} M_\Psi &= 3\,097 \text{ MeV}/c^2 \,, \\ M_{\Psi'} &= 3\,686 \text{ MeV}/c^2 \,. \end{aligned} \tag{3}$$

The experiments used synchrotons, colliding electrons and positrons moving at the same speed, but in different directions, leading (among many other possibilities!) to the two following processes:

$$\begin{aligned} e^+ + e^- &\longrightarrow \Psi \\ e^+ + e^- &\longrightarrow \Psi' \end{aligned} \tag{4}$$

The former is shown schematically in an arbitrary reference frame in Figure 1, defining the names of the associated 4-momenta.

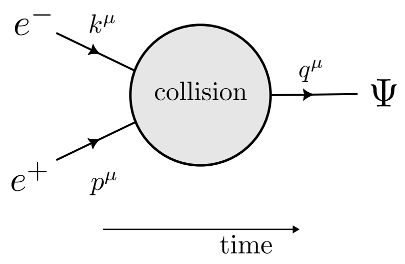

Figure 1: Collision process in an arbitrary frame.

a) Give the components of the 4-momenta $p^\mu\,, k^\mu\,, q^\mu$ in the center-of-mass reference frame. Assume the incoming particles move only along the x-axis.

b) Write the 4-momentum conservation law for the process creating a Ψ.

c) Show that at threshold, i.e. when the incoming particles have just enough velocity to create a Ψ at rest, we must have:[1]

$$\mathcal{E}_{e^-} = c^2 \frac{(M_\Psi)^2 + (m_{e^-})^2 - (m_{e^+})^2}{2M_\psi} \,, \tag{5}$$

where $\mathcal{E}_{e^-}$ is the relativistic energy of the electron.

d) Of course, positrons and electrons are anti-particles and they have the same mass, $m_{e^+} = m_{e^-} = 0.511 \text{ MeV}/c^2$. What are the minimal velocities the electron and positron must have to create the particles Ψ or Ψ'? Are they significantly different?

[1] The relations computed in Problem 1 might be useful.

e) The particle Ψ' can then disintegrate into a particle Ψ and a various number of pions $\pi^{\pm}$:[2]

$$\Psi' \quad \longrightarrow \quad \Psi + n\,\pi^+ + n\,\pi^- \tag{6}$$

Given that pions have mass $m_\pi = 140\ \mathrm{MeV}/c^2$, what is the maximal value of n?

f) Although this is an extremely rare process that has never been observed experimentally, the Ψ' could instead disintegrate into two photons (γ particles):

$$\Psi' \quad \longrightarrow \quad \gamma + \gamma\,. \tag{7}$$

What would be the smallest wavelength of the light we would detect?[3]

g) What changed from Equation (5) in this case?

h) If we had done similar computations in the framework of Newtonian Mechanics, would we have found good results (argue without doing any computations)?

[2]For simplicity, work in the rest frame of Ψ' and assume all pions are created at rest.
[3]Recall that for light(=photons), $\mathcal{E} = \frac{hc}{\lambda}$ with $hc \simeq 1.240\ \mathrm{eV} \cdot \mu\mathrm{m}$.