SPECIAL RELATIVITY (PHY-LA-SRT)
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Winter term 2022/23

PROJECT I

Due date: 20th of December 2022

- You can use any means you deem necessary to solve the problems, but your submission has to be written by you.
- Write your name and matriculation number at the beginning of each page of the document.
- Solutions can be handed in either English or German.
- Refrain from using numerical values before the end of a given question, and always clearly explain where they come from.
- Upload via Moodle by 2022-12-20, 23:59. **Only files in pdf format will be accepted, no exceptions** (no Word documents, no giant zip file with jpegs in it, . . .).
- If more convenient, it is also possible to hand in a paper version. In that case, the deadline is at the beginning of the lecture on 2022-12-20, 16:15.
- A German translation of some of the important terms can be found in the *Wortschatz* document that has been uploaded on the Moodle.

Problem 2.1 (The Train in the Tunnel Paradox):
A train approaches a tunnel at high speed $v = \frac{\sqrt{3}}{2}c$. At such a speed ($\sim 85\%$ of the speed of light), Special Relativity applies. The length at rest of the train and the tunnel are

$$L_{\text{train}}^{(\text{rest})} = 2L\,, \qquad L_{\text{tunnel}}^{(\text{rest})} = L,$$

respectively.

The conductor says that the train will not fit in the tunnel, while a worker at the entrance of the tunnel argues that there will be a precise time where the whole train will be in the tunnel. Who is right? What is the resolution of this apparent paradox? To answer this, let us first define the following inertial reference frames:

- $\mathcal{R}$: an observer at the entrance of the tunnel;
- $\mathcal{R}'$: an observer sitting at the very end of the train.

The train tracks are assumed to go in a straight line, which we defined to be the x-direction. We can therefore ignore the y- and z-directions.

a) Draw a picture of the situation before the train enters the tunnel from the point of view of $\mathcal{R}$ and $\mathcal{R}'$, clearly defining all the necessary variables.

b) What are the observed lengths of the train and the tunnel in either reference frames?

c) Draw (schematically, no need to be numerically precise) the Minkowski diagrams for both the train and the tunnel from the point of view of $\mathcal{R}$ and $\mathcal{R}'$.

   NB: Both the tunnel and the train are not points, and span a world*sheet* rather than a worldline. Therefore, draw the wordlines for points at the start and the end of the tunnel/train. Draw them large enough for both this question and the next!

d) We now define the following events:

   - **A**: The beginning of the train *enters* the tunnel;
   - **B**: The beginning of the train *exits* the tunnel;
   - **C**: The end of the train *enters* the tunnel;
   - **D**: The end of the train *exits* the tunnel.

   Indicate each of those events on the Minkowski diagrams.

   From now on, we will assume that the event **A** is the spacetime origin of both $\mathcal{R}$ and $\mathcal{R}'$.

e) Give the spacetime coordinates for all four events defined in point d) in the inertial reference frame $\mathcal{R}$.

f) Do the same for $\mathcal{R}'$. Explain how you obtained your result.

g) How much time does it take for the train to cross the tunnel, and are the events **C** and **B** simultaneous? Justify your answer in both reference frames.

h) Compute the spacetime interval in both frames.

i) Who was wrong, or is there a paradox at all? Why, and which feature of Special Relativity are you using to justify your answer?

Problem 2.2 (Binary Systems and Relativistic Jets):
SS 433 is one of the most exotic stellar system that we have found in the Milky Way. It is a binary system, namely one that does not have a single star, but in this case a star and a black hole. Matter is sucked out of the star by gravitational forces, and slowly falls into the black hole to form what is called an **accretion disk**.

Moreover, through a process far beyond the scope of this course, part of this matter is ejected into tight beams at high speed along the poles of the black holes. These beams are called **astrophysical jets**, and an artist rendition of SS 433 is depicted in Figure 1. This system is one of many celestial objects studied by the astrophysics group at Uni-Hamburg/DESY, and additional informations including animations can be found at the link below.[1]

The goal of this exercise is—through guided steps—to determine the principal features of SS 433 using what we have seen during the lecture.

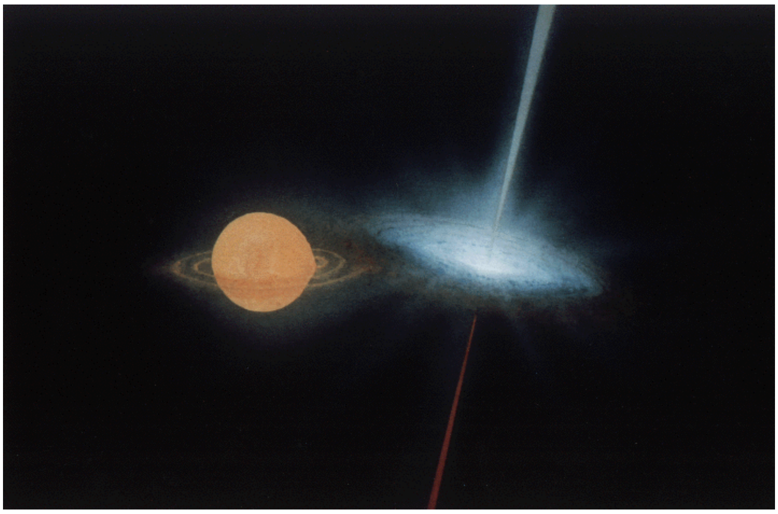

Figure 1: Artist rendition of SS 433 [NASA]. Part of the star is pulled by gravitational forces and forms an accretion disk around the black hole (right). The two beams appearing from the black hole are the astrophysical jets.

The matter in SS 433's star is mainly constituted of hydrogen atoms, the most common element in the Universe. This can be used to determine many of the properties of the jets. Indeed, as we have seen in the lecture, according to Bohr's model atoms emits light at precise wavelengths, λ. For hydrogen, we obtain the **Hydrogen emission spectrum** shown in Figure 2. We only show the so-called H-sequence, which is emitted in the visible spectrum. For our purpose, we can focus on the H-α line:

$$\text{H-}\alpha: \qquad \lambda = 6563\text{Å} = 656.3\,\text{nm}$$

In 1979, *Margon et al.* obtained the emission spectrum of SS 433, using instrumentations at the Lick Observatory (San Diego, California).[2] Part of their results is given in Figure 3, showing emission spectra for three consecutive days.

[1] `https://www.desy.de/aktuelles/news_suche/index_ger.html?openDirectAnchor=1887`
[2] Margon, B., et al. *"The bizarre spectrum of SS 433."* The Astrophysical Journal **230** (1979): L41–L45.

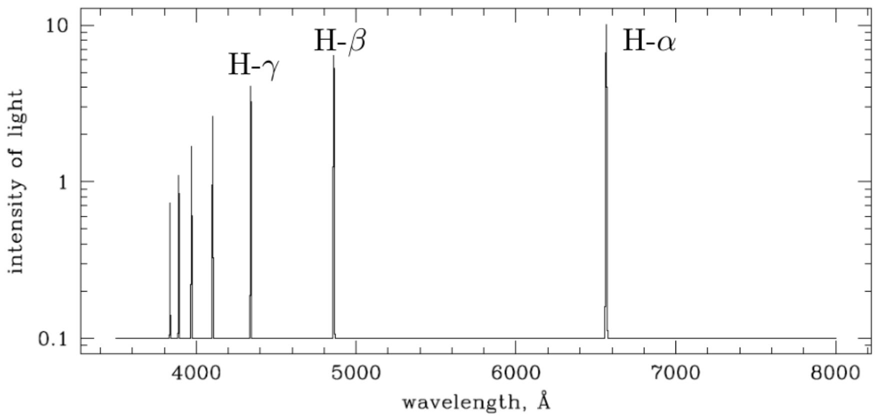

Figure 2: Emission Spectrum of Hydrogen. Only the visible part of the electromagnetic spectrum is shown.

Due to noise from other sources and experimental imprecisions, the spectra are not as clean as those found in experiments made on Earth, but we can make out three distinct peaks. It turns out that they all correspond to H-α lines. (H-α)$_2$ is the emission line for matter in the accretion disk, while the other two are associated with hydrogen in jets.

a) None of these lines are at precisely $\lambda = 6563\,\text{Å}$, due to the relativistic Doppler effect. In a few words, describe why this is the case, and what is the main difference between the usual Doppler effect and the relativistic version.

b) For a source moving with a velocity $\vec{v}$ emitting light at a wavelength λ, the wavelength observed by someone at rest, $\lambda_{\text{obs.}}$ is given by

$$\lambda_{\text{osb}} = \lambda \sqrt{\frac{1 + \vec{\beta} \cdot \vec{n}}{1 - \vec{\beta} \cdot \vec{n}}},$$

where $\vec{\beta} = \vec{v}/c$ and $\vec{n}$ is the unit vector in the observer's frame directed from the observer to the source, and satisfies $|\vec{n}| = 1$.

In the special case where $\vec{n} = (1, 0, 0)$, draw a Minkowski diagram and rederive this formula.

c) From Figure 3, argue that SS 433 moves very slowly with respect to Earth. What peak(s) did you use to make this observations?

d) From the previous question, we can assume that SS 433 is at fixed distance from Earth. We will moreover assume that the jets are symmetric—at any given time, matter is ejected from the black hole at the same velocity in exactly opposite directions. The situation is summarised in Figure 4.

Using any method you deem appropriate to have a precision of about 20Å for the observed wavelengths (A ruler might be useful...), determine the velocity of particles in the jet. Which peaks are you using to make your estimation? For reference, *Margon et al.* find that $|\vec{v}| \simeq 0.26c$.

e) From Figure 3, argue without doing any computations that the jets are not stationary, but are *precessing*: they rotate along an axis. Can you deduce the direction of rotation?

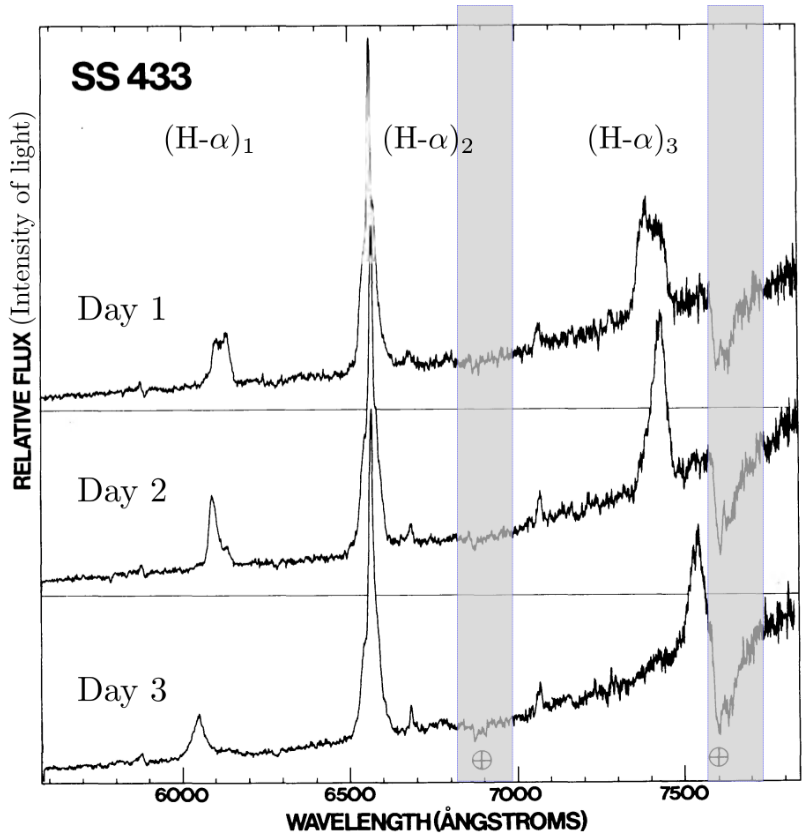

Figure 3: Emission spectrum from SS 433 on three consecutive days. All three spikes corresponds to H-α lines. The grey bands are bad data and can be ignored.

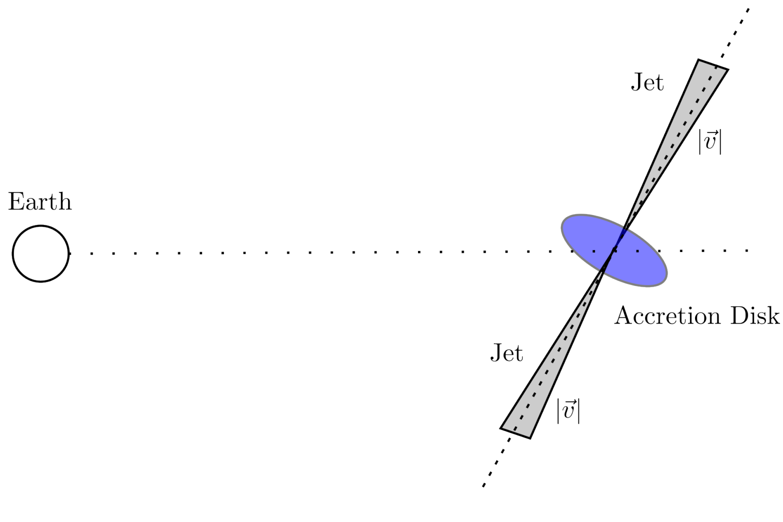

Figure 4: Schematic description of SS 433, and an observation from Earth.