# 9) Gravitational waves

Solving Einstein : HARD

Easy : $g_{\mu\nu} = \eta_{\mu\nu}$

expand perturbations around something known

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

inverse : $$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

NB: $h_{\mu\nu} h^{\rho\sigma}$ subleading $\rightarrow$ ignored

$\rightarrow$ raise/lower indices w/ $\eta$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} \partial_\rho \partial_\nu h_{\mu\sigma} \pm (\text{perms}) + O(h^2)$$

$$R = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h \qquad h = \eta^{\mu\nu} h_{\mu\nu}$$

Einstein equations

$$G_{\mu\nu} = \frac{8\pi}{c^4} G_N T_{\mu\nu}$$

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu} \Box h\right)$$

→ A priori very hard!

NB: we have gauge d.o.f. (diff)

$$x^\mu \to \tilde{x}^\mu = x^\mu + \varepsilon^\mu \qquad |\varepsilon| \ll 1$$

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}$$

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu \qquad (9.6)$$

$$\tilde{h}^{\mu\nu} = h^{\mu\nu} - \partial^\mu \varepsilon^\nu - \partial^\nu \varepsilon^\mu$$

If we use "good" frame, maybe equations are simpler

Detour: Maxwell (in flat space)

gauge field $A_\mu = (\phi, \vec{A})$

field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

4-current: $j_\mu = (\rho, \vec{J})$

inhom. Maxwell $\partial^\mu F_{\mu\nu} = j_\nu$ (sloppy w/ units)

Maxwell invariant under gauge transform.

$$A_\mu \xrightarrow{\text{gauge}} \tilde{A}_\mu = A_\mu + \partial_\mu \alpha \qquad \alpha = \alpha(x) \text{ fct}$$

$$F_{\mu\nu} \to \tilde{F}_{\mu\nu} = \partial_\mu A_\nu + \cancel{\partial_\mu \partial_\nu \alpha} - \partial_\nu A_\mu + \cancel{\partial_\nu \partial_\mu \alpha}$$
$$= F_{\mu\nu}$$

in terms of gauge pot

$$\partial^\mu F_{\mu\nu} = \Box A_\nu - \partial_\nu(\partial_\mu A^\mu) = j_\nu$$

using gauge transfo, can find

$$\partial_\alpha A^\alpha = 0 \iff \Box\alpha = -\partial_\mu A^\mu$$

such $\alpha$ is a gauge choice and simplify Maxwell.

choice $\partial_\mu A^\mu = 0$ → Lorentz gauge

$$\partial^\mu F_{\mu\nu} = \Box A_\nu - \partial_\nu \cancel{(\partial_\mu A^\mu)}^{\,0} = j_\nu$$

$$\rightarrow \quad \Box A_\nu = j_\nu \quad (*)$$

since $F_{\mu\nu}$ is gauge inv, sol for A give general sol.

goal: do same for $G_{\mu\nu}$ to obtain simpler equations using diffs instead of gauge

Let us introduce

$$D_\nu = \partial_\mu h^\mu{}_\nu - \frac{1}{2} \partial_\nu h^\mu_\mu \qquad (9.13)$$

using $\tilde{h}_{\mu\nu}$ (9.6)

$$\partial_\mu \tilde{h}^\mu{}_\nu = \partial_\mu (h^\mu_\nu - \partial^\mu \varepsilon_\nu - \partial_\nu \varepsilon^\mu)$$

$$= \partial_\mu h^\mu_\nu - \Box \varepsilon_\nu - \partial_\mu \partial_\nu \varepsilon^\mu$$

$$\partial_\nu \tilde{h}^\mu_\mu = \partial_\nu h^\mu_\mu - 2 \partial_\mu \partial_\nu \varepsilon^\mu$$

$$\Rightarrow \tilde{D}_\nu = \partial_\mu \tilde{h}^\mu{}_\nu - \frac{1}{2} \partial_\nu \tilde{h}^\mu_\mu$$

$$\tilde{D}_\nu = \partial_\mu h^\mu{}_\nu - \Box\varepsilon_\nu - \cancel{\partial_\mu\partial_\nu\varepsilon^\mu}$$

$$-\frac{1}{2}\left(\partial_\nu h^\mu{}_\mu - \cancel{2\partial_\mu\partial_\nu\varepsilon^\mu}\right)$$

$$\overset{(9.13)}{=} D_\nu - \Box\varepsilon_\nu \qquad (9.14)$$

Under diff, the combination shifts by $\Box\varepsilon$. We can choose $\varepsilon_\mu$ such that

$$\tilde{D}_\nu = 0$$

need to solve

$$\Box\varepsilon_\nu = D_\nu = \partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h^\mu{}_\mu$$

de Donder gauge: $D_\nu = 0$

(same as Lorenz gauge for EM)

physically, de Donder gauge is a choice of coordinates where $D_\nu = 0$

NB: in practice solving PDE is hard, but the solution exists → no need for explicit formula

At linear order

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\nu D_\mu + \partial_\nu D_\mu - \Box h_{\mu\nu}\right)$$

in de Donder gauge

$$R_{\mu\nu} = - \Box h_{\mu\nu} \quad !$$

A lot simpler!

In vacuum, $T_{\mu\nu} = 0 \Rightarrow G_{\mu\nu} = 0$

$\Rightarrow R_{\mu\nu} = 0$

At linearised level

$$\Box h_{\mu\nu} = 0 \qquad (9.19)$$

This is a wave equation for the 16 components of the metric perturbation!

solutions using ansatz

$$h_{\mu\nu}(x) = \varepsilon_{\mu\nu}\, e^{i(\omega t - \vec{k}\cdot\vec{x})}$$

$$= \varepsilon_{\mu\nu}\, e^{i k_\beta x^\beta}$$

$$k^\nu = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix} \quad \text{wave vector} \qquad \partial_\mu h_{\nu\beta} = -i k_\mu h_{\nu\beta}$$

$\varepsilon_{\mu\nu}$ amplitude

$$\Box h_{\mu\nu} = \varepsilon_{\mu\nu}\, \partial_\alpha \partial^\alpha e^{i k_\mu x^\mu} = \varepsilon_{\mu\nu} (-i)^2 k_\alpha k^\alpha e^{i k_\beta x^\beta}$$

$\rightarrow k_\mu$ is light like

We are in the de Donder gauge

$$0 \stackrel{!}{=} D_\nu = \partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h^\mu{}_\mu$$

$$= -ik_\mu h^\mu{}_\nu - \frac{1}{2}(-i)k_\nu h^\mu{}_\mu$$

$$= -i\left(k_\mu \varepsilon^\mu{}_\nu - \frac{1}{2}k_\nu \varepsilon^\mu{}_\mu\right)e^{ik_\alpha x^\alpha}$$

$\Rightarrow$ $k_\mu \varepsilon^\mu{}_\nu = \frac{1}{2}k_\nu \varepsilon^\mu{}_\mu$ in de Donder gauge

(remember $\varepsilon^\mu{}_\nu = \eta^{\mu\alpha}\varepsilon_{\alpha\nu}$ at linear order)

choose $k^\mu = \begin{pmatrix} k \\ +k \end{pmatrix}$ $\quad k_\mu = \begin{pmatrix} k \\ -k \end{pmatrix}$

$$2\eta^{\mu\rho}\varepsilon_{\rho\nu}k_\mu = \eta^{\mu\rho}\varepsilon_{\rho\mu}k_\nu$$

$\underline{\nu = 0}$ : $\quad 2(e_{00} + e_{30})k = (e_{00} - e_{11} - e_{22} - e_{33})k$

$\nu = 1 \quad 2(e_{01} + e_{31})k = 0$

$\nu = 2 \quad 2(e_{02} + e_{32})k = 0$

$\nu = 3 \quad 2(e_{03} + e_{33})k = -(e_{00} - e_{11} - e_{22} - e_{33})k$

$g_{\mu\nu}$ sym $\Rightarrow h_{\mu\nu}$ sym $\Rightarrow \varepsilon_{\mu\nu}$ sym

<u>solutions:</u>
$$e_{01} = -e_{31}$$
$$e_{02} = -e_{32}$$
$$2e_{30} + e_{00} + e_{33} = 0$$
$$e_{22} = -e_{11}$$

$e_{\mu\nu}$ : 10 dof (sym) $-$ 4 constraints

$\rightarrow$ 6 free param

diff for de Donder

$$\Box \varepsilon_\nu = D_\nu = 0$$

Ansatz $\varepsilon_\nu = a_\nu e^{i k_\alpha x^\alpha}$ $\left(k^\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ -k \end{pmatrix}\right)$

$$\Box \varepsilon_\nu = -\underbrace{k_\alpha k^\alpha}_{=0} \varepsilon_\nu = 0 \quad \checkmark$$

→ any diff $\varepsilon_\nu = a_\nu e^{i k_\alpha x^\alpha}$ leads to de Donder gauge

Moreover if

$$h_{\mu\nu} \xrightarrow{\text{diff}} h_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu$$

$$\epsilon_{\mu\nu} \xrightarrow{\text{diff}} \epsilon_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu$$

$$= \epsilon_{\mu\nu} - (i k_\mu a_\nu) - (i k_\nu a_\mu)$$

We can use diff to not only go to de Donder gauge, but also "gauge away" more $\varepsilon_{\mu\nu}$ components by going to even more specific choice of coord by setting $a_\nu \in \mathbb{C}$

$$\tilde{e}_{00} = e_{00} - 2ik\, a_0 \overset{!}{=} 0$$

$$\tilde{e}_{11} = e_{11}$$

$$\tilde{e}_{33} = e_{33} + 2ik\, a_3 \overset{!}{=} 0$$

$$\tilde{e}_{12} = e_{12}$$

$$\tilde{e}_{13} = e_{13} + ik\, a_1 \overset{!}{=} 0$$

$$\tilde{e}_{23} = e_{23} + ik\, a_2 \overset{!}{=} 0$$

Go to a frame where

$$\tilde{e}_{12} = \tilde{e}_{21} \overset{!}{=} e_\times \qquad \tilde{e}_{11} = -\tilde{e}_{22} = e_+$$

$e_{\mu\nu} = 0$ else

$$\tilde{e}_{\mu\nu} = e_+ \overbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}^{h^{(+)}} + e_\times \overbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}^{h^{(\times)}}$$

$$h_{\mu\nu}(x,k) = \left(e_+ h^{(+)}_{\mu\nu} + e_\times h^{(\times)}_{\mu\nu}\right) e^{i k_\alpha x^\alpha}$$

$$k_\alpha = (k, 0, 0, -k)$$

# Summary

$$G_{\mu\nu} = 0 \quad \text{(Einstein in vacuum)}$$

$$\Big\downarrow g = \eta + h$$

$$-\Box h_{\mu\nu} + \partial_\mu D_\nu + \partial_\nu D_\mu = 0$$

$$\Big\downarrow x \longrightarrow x + \varepsilon, \; \varepsilon_\mu = a_\mu e^{ikx} \Rightarrow \Box \varepsilon_\mu = D_\nu \overset{!}{=} 0$$

$$-\Box h_{\mu\nu} = 0 \quad \rightarrow$$

particular sol $h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik^\alpha x_\alpha}$ $k_\alpha = (k, 0, 0, k)$

$\Rightarrow$ grav wave: part of metric propagates like wave in vacuum

dof: $10$ ($h_{\mu\nu} = h_{\nu\mu}$) $-\ 4$ ($\Box \varepsilon_\mu = 0$) $-\ 4$ ($a_\nu$) $= 2$

| | EM | GR |
|---|---|---|
| vacuum | $\partial_\mu F^{\mu\nu} = 0$ | $G_{\mu\nu} = 0$ |
| gauge | $\partial_\mu A^\mu = 0$ | $D_\nu = 0$ |
| EOM | $\Box A_\nu = 0$ | $\Box h_{\mu\nu} = 0$ |

both cases

source