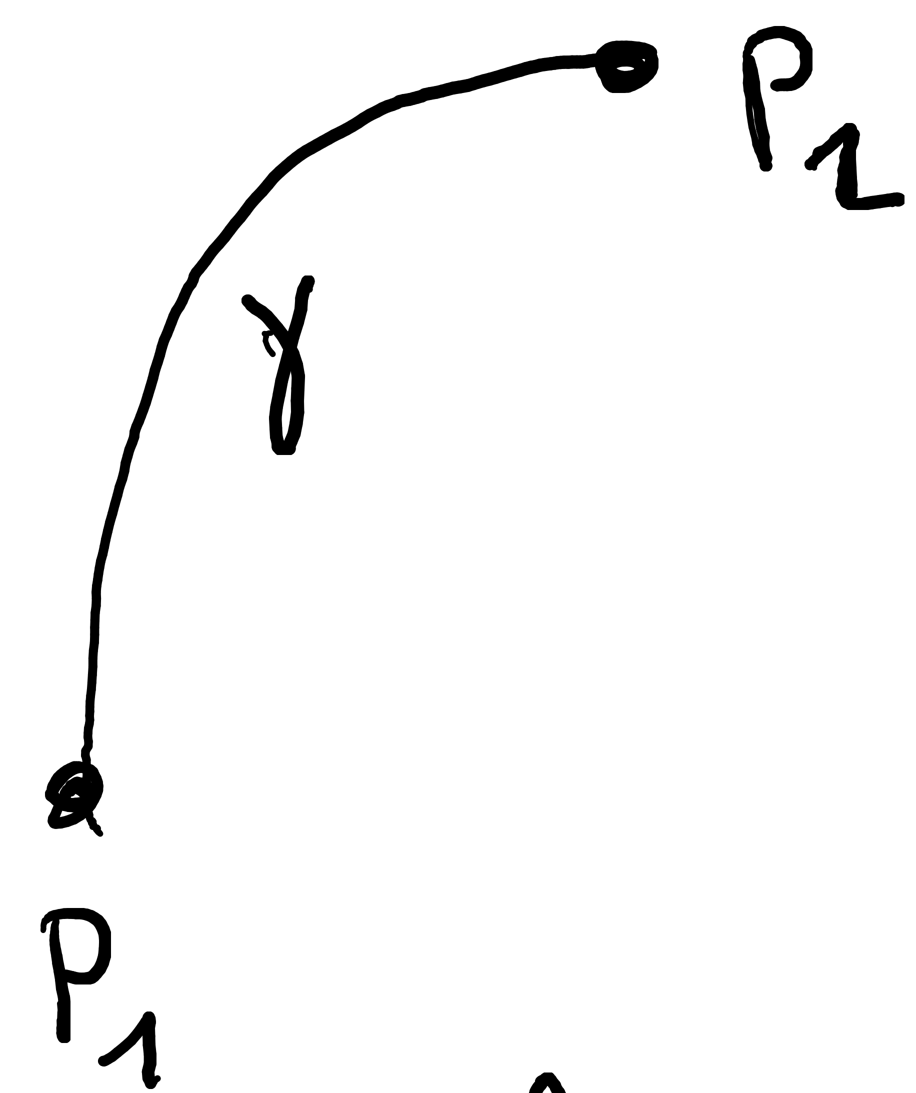
Blav GR unibe ⚠ $\eta = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$ sec 2.5

Curves and Geodesics



$$\gamma : [a,b] \longrightarrow \mathcal{M}$$
$$\lambda \longmapsto x^\mu(\lambda)$$
$$x^\mu(a) : P_1 \qquad x^\mu(b) : P_2$$

$$l(\gamma) = \int_\gamma ds = \int_a^b d\lambda \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

What paths extremise the distance?

Let us do a small variation

$$\tilde{\gamma} : \lambda \longrightarrow x^\mu(\lambda) + \varepsilon\, \delta x^\mu(\lambda)$$

boundary $\delta x^\mu(a) = 0 = \delta x^\mu(b)$

This is a Lagrangian description

$$\ell = \int_a^b d\lambda \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \leftrightarrow S = \int_{t_i}^{t_f} dt\, L[\vec{x}, \dot{\vec{x}}]$$

We look for extrema

$$x^\mu \to x^\mu + \delta x^\mu$$

$$L[x^\mu + \delta x^\mu, \dot{x} + \delta\dot{x}] = L[x^\mu, \dot{x}^\mu] + \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\delta \dot{x}^\mu$$

$$\Rightarrow \delta S = \int_a^b \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\delta \frac{dx^\mu}{d\lambda}$$

using $\delta \frac{d}{d\lambda} = \frac{d}{d\lambda}\delta$ + IbP

$$\left(\begin{array}{l} \delta x = x_1 - x_2 \\ \delta \dot{x} = \dot{x}_1 - \dot{x}_2 = \frac{d}{d\lambda}(x_1 - x_2) = \frac{d}{d\lambda}\delta x \end{array} \right)$$

$$\delta S = \int_a^b \left(\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu} \right) \delta x^\mu$$

extremum $\delta S = 0 \Leftrightarrow \frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu} = 0$

(Euler-Lagrange)

apply to $L = \frac{1}{2} g_{\mu\nu}(x)\, \dot{x}^{\mu} \dot{x}^{\nu}$

$$\frac{\partial L}{\partial x^{\lambda}} = \frac{1}{2}\partial_{\lambda} g_{\mu\nu}\, \dot{x}^{\mu} \dot{x}^{\nu} = \frac{1}{2} g_{\lambda\alpha} g^{\alpha\beta} \partial_{\beta} g_{\mu\nu}\, \dot{x}^{\mu} \dot{x}^{\nu}$$

$$\frac{\partial L}{\partial \dot{x}^{\lambda}} = \frac{1}{2} g_{\mu\nu}\left(\delta^{\mu}_{\lambda} \dot{x}^{\nu} + \dot{x}^{\mu} \delta^{\nu}_{\lambda}\right)$$

$$= \frac{2}{2} g_{\lambda\nu} \dot{x}^{\nu}$$

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^{\lambda}} = \partial_{\mu} g_{\lambda\nu}\, \dot{x}^{\nu} \dot{x}^{\mu} + g_{\lambda\nu}\, \ddot{x}^{\nu}$$

$$= g_{\lambda\alpha}\left(\ddot{x}^{\alpha} + g^{\alpha\beta} \partial_{\mu} g_{\alpha\nu}\, \dot{x}^{\mu} \dot{x}^{\nu}\right)$$

EL $0 = g_{\lambda\alpha}\left(\ddot{x}^{\alpha} + g^{\alpha\beta}\left(\partial_{\mu} g_{\alpha\nu} - \partial_{\beta} g_{\mu\nu}\right) \dot{x}^{\mu} \dot{x}^{\nu}\right)$

$\Leftrightarrow 0 = \ddot{x}^{\alpha} + \Gamma^{\alpha}_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = 0$!!!

Free-falling object extremise interval

NB: γ is param indep $s \to f(\lambda)$ (see exercise)

$\to \frac{dL}{d\lambda} = 0$ same as $\frac{dL}{dt} = 0$ in stat mech

replace $L \to L^2$ in action

$$(EL) + \cancel{\frac{dL}{d\lambda}}^{\,0}(\ldots) = 0$$

curves $\gamma \subset M$ extremising the distance are called <u>geodesics</u>

5) Curvatures

Note: In GR we want covariant expressions
Answers should not explicitely depend
on $\partial_\mu = \frac{\partial}{\partial x^\mu}$ but rather ∇_μ

However $[\partial_\mu, \partial_\nu] = 0$

what about covariant derivatives?

$$[\nabla_\mu, \nabla_\nu] B^\rho = \nabla_\mu \nabla_\nu B^\rho - \nabla_\nu \nabla_\mu B^\rho = 2\nabla_{[\mu}\nabla_{\nu]} B^\rho$$

$$= \nabla_\mu (\partial_\nu B^\rho + \Gamma^\rho_{\nu\alpha} B^\alpha)$$

$$- \nabla_\nu (\partial_\mu B^\rho - \Gamma^\rho_{\mu\alpha} B^\alpha)$$

Note: $\nabla \partial B$ $\nabla \Gamma B$ are not covariant

For a 2-tensor $C_\alpha{}^\beta$

$$\nabla_\mu C_\alpha{}^\beta = \partial_\mu C_\alpha{}^\beta + \Gamma^\beta_{\mu\gamma} C_\alpha{}^\gamma - \Gamma^\gamma_{\mu\alpha} C_\gamma{}^\beta$$

$$\nabla_\mu \partial_\nu B^\rho = \underbrace{\partial_\mu \partial_\nu B^\rho}_{\text{sym}} + \Gamma^\rho_{\mu\alpha} \partial_\nu B^\alpha - \underbrace{\Gamma^\alpha_{\mu\nu} \partial_\alpha B^\rho}_{\text{sym}}$$

$$\nabla_{[\mu} \partial_{\nu]} B^\rho = \Gamma^\rho_{\alpha[\mu} \partial_{\nu]} B^\alpha$$

$$\nabla_\mu \Gamma^\rho_{\nu\alpha} B^\alpha = \partial_\mu (\Gamma^\rho_{\nu\alpha} B^\alpha) + \Gamma^\rho_{\mu\beta} \Gamma^\beta_{\nu\alpha} B^\alpha - \underbrace{\Gamma^\beta_{\mu\nu} \Gamma^\rho_{\beta\alpha} B^\alpha}_{\text{sym}}$$

$$\nabla_{[\mu} \Gamma^\rho_{\nu]\alpha} B^\alpha = \partial_{[\mu} \Gamma^\rho_{\nu]\alpha} B^\alpha - \Gamma_{\alpha[\mu} \partial_{\nu]} B^\alpha + \Gamma^\rho_{\beta[\mu} \Gamma^\beta_{\nu]\alpha} B^\alpha$$

$$\Rightarrow \nabla_{[\mu} \nabla_{\nu]} B^\rho = \left(\partial_{[\mu} \Gamma^\rho_{\nu]\alpha} + \Gamma^\rho_{\beta[\mu} \Gamma^\beta_{\nu]\alpha} \right) B^\alpha$$

We therefore have

$$[\nabla_\mu, \nabla_\nu] B^\rho = R^\rho{}_{\lambda\mu\nu} B^\lambda$$

Riemann tensor

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}$$

NB: LHS covariant by constr

RHS not covariant term by term

Useful properties

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda{}_{\nu\rho\sigma}$$

One can show

$g_{\mu\nu,\rho\sigma} = \partial_\rho \partial_\sigma g_{\mu\nu}$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma,\nu\rho} - g_{\mu\rho,\nu\sigma} + g_{\rho\nu,\mu\sigma} - g_{\sigma\nu,\mu\rho}\right) + g_{\lambda\tau}\left(\Gamma^\tau_{\mu\sigma}\Gamma^\lambda_{\rho\nu} - \Gamma^\tau_{\mu\rho}\Gamma^\lambda_{\sigma\nu}\right)$$

From there using $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu \cdot \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$

$$R_{\mu\nu\sigma\tau} = - R_{\nu\mu\sigma\tau} \qquad \text{(asym 1-2)}$$

$$R_{\mu\nu\sigma\tau} = - R_{\mu\nu\tau\sigma} \qquad \text{(asym 34)}$$

$$R_{\mu\nu\sigma\tau} = R_{\sigma\tau\mu\nu} \qquad \text{(sym (12)} \leftrightarrow \text{(34))}$$

cyclicity $R_{\mu\nu\sigma\rho} + R_{\mu\sigma\rho\nu} + R_{\mu\rho\nu\sigma} = 0$

Bianchi identity $\nabla_\mu R_{\nu\rho\sigma\tau} + \nabla_\rho R_{\mu\nu\sigma\tau} + \nabla_\nu R_{\rho\mu\sigma\tau} = 0$

Riemann tensor encodes curvature

associated tensors:

Ricci tensor: $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$

$$R_{\mu\nu} = R_{\nu\mu}$$

Ricci scalar: $R = g^{\mu\nu} R_{\mu\nu}$

(Anti) Symmetrisation

let a tensor $B_{\mu\nu}$ can always be writen

$$B_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu}) + \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$$

$$\stackrel{!}{=} \quad S_{\mu\nu} \quad + \quad A_{\mu\nu}$$

$$S_{\mu\nu} = + S_{\nu\mu} \qquad A_{\mu\nu} = -A_{\nu\mu}$$

notation $B_{(\mu\nu)} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu})$

$$B_{[\mu\nu]} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$$

Note $B^{[\mu\nu]} C_{(\mu\nu)} = 0$

in this notation $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}(2\partial_{(\mu} g_{\nu)\alpha} - \partial_\alpha g_{\mu\nu})$