Symmetries

classical mech

Noether: if transformation $\vec{x} \to \vec{x}'$

st $L(\vec{x}', \dot{\vec{x}}') = L(\vec{x}, \dot{\vec{x}})$

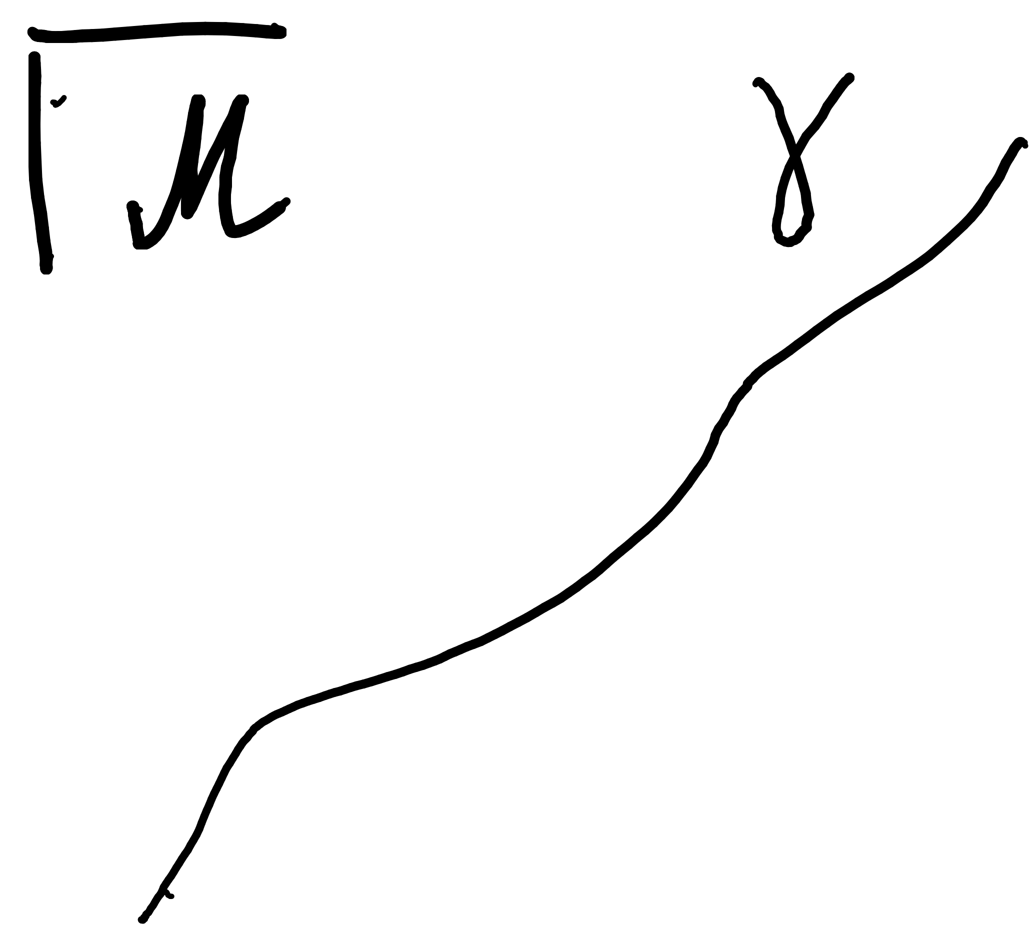
$\exists Q$ such that $\dot{Q} = \frac{d}{dt} Q = 0$

example : $\vec{r} \to \vec{r} + \delta\theta \, \vec{n} \wedge \vec{r}$

$\vec{Q} = \vec{L} = \vec{r} \wedge \vec{p}$

angular momentum is conserved

Killing Vectors



$\gamma : [a,b] \to M$

$\lambda \to x^\nu(\lambda)$

if γ geodesic $D_\lambda \dot{x}^\mu = \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0$

scalar

Let $Q = K_\mu \dot{x}^\mu$

Along curve γ

$$D_\lambda Q = \frac{d}{d\lambda} Q = (D_\lambda K_\mu)\dot{x}^\mu + K_\mu \overbrace{(D_\lambda \dot{x}^\mu)}^{=0 \text{ (geod}}$$

$$= \nabla_\nu K_\mu \dot{x}^\mu \dot{x}^\nu = \frac{1}{2}(\nabla_\nu K_\mu + \nabla_\mu K_\nu)\dot{x}^\mu \dot{x}^\nu$$

$$= \frac{1}{2}(\mathcal{L}_K g_{\mu\nu})\dot{x}^\mu \dot{x}^\nu$$

if K a killing vector Q conserved along γ!

Under diff

$$x^\mu \to x^\mu + \xi^\mu$$

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} \; g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$$

symmetry if $\mathcal{L}_\xi g_{\mu\nu} = 0$

$g_{\mu\nu}$ symmetric : $\frac{d(d+1)}{2}$ equations

in $d=3$: max 6 indep solutions (depends on manifold)

in $\mathbb{R}^3$: 3 translations

3 rotations

in $\mathbb{R}^{1,3}$: 3 translations

3 rot

3 boosts

in T^2 : only one killing vector !

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$$\mathcal{L}_K g_{ij} = K^k \partial_k g_{ij} + g_{ik} \partial_j K^k + g_{kj} \partial_i K^k \overset{!}{=} 0$$

· Note g = cst $\partial_i g_{jk} = 0$

in index-free notation

$$K = K^i \partial_i$$

$$\mathcal{L}_K g_{ij} = K(g_{ij}) + g_{ik} \partial_j K^k + g_{kj} \partial_i K^k$$

if coord system x^i such that $\partial_i g_{jk} = 0$

and K^k cst (in that frame)

$\rightarrow$ K is Killing

e)

in Cartesian coord

$\mathbb{R}^3$: $ds^2 = dx^2 + dy^2 + dz^2 = g_{ij}\, dx^i dx^j$

$x^1 = x \quad x^2 = y \quad x^3 = z$

$g_{ij} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

clearly $\partial_k g_{ij} = 0$

$K = K^k \partial_k$ is killing if K^k cst

3 independent killing vectors

$X^i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad Y^i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad Z^i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

2) In polar coordinates

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin\theta \, d\varphi^2$$

NB: $ds^2_{(x,y,z)} = ds^2_{(r,\theta,\varphi)}$

note $\partial_\varphi g_{\mu\nu} = 0$ in this coord system

in this frame

$R = 1 \partial_\varphi$ is killing vector

back to Euclidean

$$\partial_\varphi = \frac{\partial x^i}{\partial \varphi} \frac{\partial}{\partial x^i} = -y \partial_x + x \partial_y$$

$$\mathcal{L}_R g_{ij} = R^k \partial_k g_{ij} + g_{ik}\partial_j R^k + g_{kj}\partial_i R^k$$

$$\mathcal{L}_R g_{12} = 0 + g_{1k}\partial_2 R^k + g_{k2}\partial_1 R^k$$

$$= \partial_2 R^1 + \partial_1 R^2 = -1 + 1$$

Similarly

$$R = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \qquad S = \begin{pmatrix} z \\ 0 \\ -y \end{pmatrix} \qquad T = \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}$$

are also killing vectors

R rotation along Z (∂_φ)

S " " Y

T " " X

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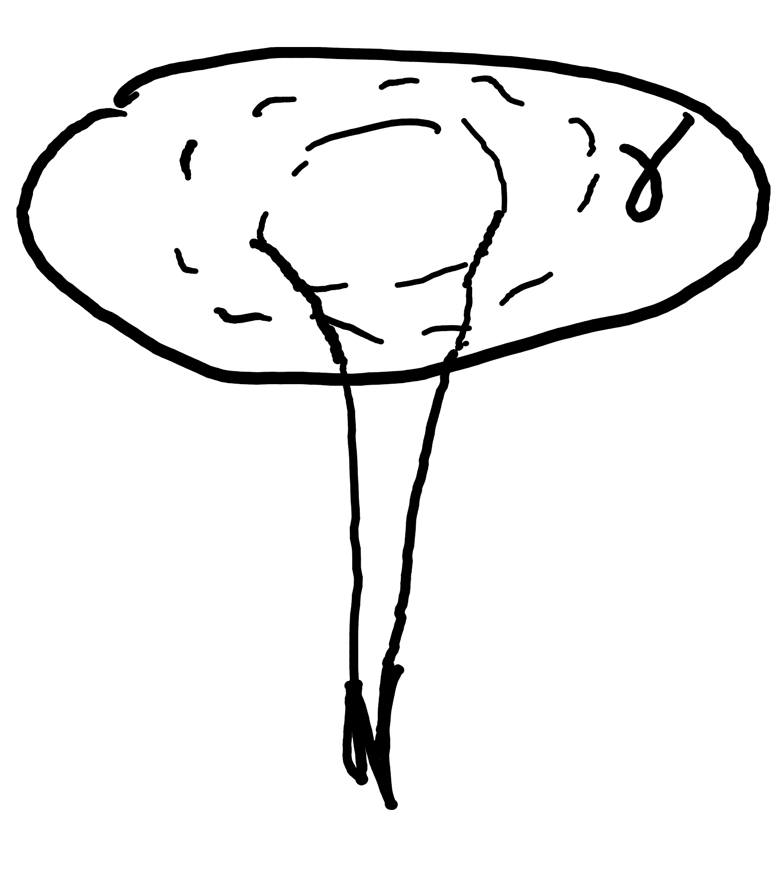
Consider particle of mass m

spherical symmetry $\quad A = \frac{1}{B} = 1 - \frac{2MG_N}{r}$

$$ds^2 = A\,dt^2 - \frac{1}{A}\,dr^2 - d\theta^2 - \sin^2\theta\,d\varphi^2$$

Geodesics:

$$\begin{aligned} \gamma : [a,b] &\longrightarrow M \\ \lambda &\longmapsto x^\mu(\lambda) \end{aligned}$$



$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta = 0$$

look for circle ($r = r_0$, $\theta = cst = \frac{\pi}{2}$)
of constant angular velocity
($\omega = \frac{d\varphi}{ds} = cst$)

Christoffel symbols

$$\Gamma^t_{tr} = \frac{A'}{2A}$$

$$\Gamma^r_{rr} = \frac{B'}{2B} \qquad \Gamma^r_{tt} = \frac{A'}{2B} \qquad \Gamma^r_{\theta\theta} = -\frac{r}{B} \qquad \Gamma^r_{\varphi\varphi} = -\frac{r\sin^2\theta}{B}$$

$$\Gamma^\theta_{\theta r} = \frac{1}{r} \qquad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta = 0$$

$$\Gamma^\varphi_{\varphi r} = \frac{1}{r} \qquad \Gamma^\varphi_{\varphi\theta} = \cot\theta = 0$$

Brute force solution

$$\mu = t: \quad \ddot{x}^0 = 0 \overset{!}{=} \Gamma^t_{tr}\,\dot{t}\,\underbrace{\dot{r}}_{=0} = 0 \Rightarrow \alpha\lambda + \beta = t$$

$$\mu = r \quad \ddot{x}^1 = 0 \overset{!}{=} \cancel{\Gamma^r_{rr}\dot{r}\dot{r}} + \Gamma^r_{tt}\dot{t}\dot{t} + \Gamma^r_{\theta\theta}\cancel{\dot{\theta}}\dot{\theta} + \Gamma^r_{\varphi\varphi}\dot{\varphi}\dot{\varphi}$$

$$= \frac{A'}{2B}\alpha^2 - \frac{r\sin^2\theta}{B}\omega^2 \quad \circledast$$

$$\mu = \theta \quad \ddot{\theta} = 0 \overset{!}{=} \Gamma^\theta_{\theta r}\cancel{\dot{\theta}}\dot{\theta} + \cancel{\Gamma^\theta_{\varphi\varphi}}\dot{\varphi}\dot{\varphi} = 0 \quad \checkmark$$

$\gamma = \varphi$: $\quad \ddot{\varphi} = 0 \overset{!}{=} \Gamma^{\varphi}_{\varphi r} \dot{\varphi} \cancel{\dot{r}} + 2\Gamma^{\varphi}_{\cancel{\varphi\theta}} \cancel{\dot{\varphi}\dot{\theta}}$

$$0 = \frac{A'}{2B}\alpha^2 - \frac{r\sin^2\theta}{B}\omega^2 \quad (*)$$

$$\Rightarrow 0 = \frac{A'}{2B}\alpha^2 - \frac{r}{B}\omega^2$$

$$\left.\begin{aligned} A'\alpha^2 &= 2r\omega^2 \\ A' &= \frac{2MG}{r^2} \end{aligned}\right\} \Rightarrow \alpha^2 = \frac{4r_0^3}{MG}\omega^2$$

$$x^\mu = \begin{pmatrix} \alpha\lambda \\ r_0 \\ \pi/2 \\ \omega\lambda \end{pmatrix} \qquad g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \alpha^2 A - r_0\omega^2$$

if time-like geodesic

if not 1, can always normalise

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$$

$$\alpha^2 A - r_0\omega^2 = \omega^2\left(\frac{4r_0^3}{MG}A - r_0\right) \overset{!}{=} 1$$

$$\omega^2 = \frac{GM}{r_0^3}\left(1 - \frac{6GM}{2r_0}\right)^{-1}$$

if $r_0 \gg 2GM$ $\quad \omega^2 \sim \frac{GM}{r_0^3}$

in classical limit (very far away from source)
we recover Kepler's 3rd law

orbits in GR are circular to 1st approx

if light-like geodesic

$$g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 = \kappa^2 A - r_0^2 \omega^2$$

$r_0 = \frac{3}{2} r_s = 3GM$ photon sphere (circle)

light rays can be bent so much they orbit around a black hole !!!

if photon is emited at $r_0 = 3/2\, r_s$ it will orbit forever (at least classically).

NB: no free-falling object can reach the sphere

<u>corollary</u>: light can be bent