Assignment 8

Lie derivatives

under infinitesimal diff

$$x^\mu \to \tilde{x}^\mu = x^\mu + \varepsilon A^\mu$$

$$T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} \longrightarrow \tilde{T}^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_p} = \frac{\partial \tilde{x}}{\partial x} \cdots T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q}$$

Lie derivative gives

$$\mathcal{L}_V T = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(T(x) - \tilde{T}(x) \right)$$

$$\mathcal{L}_V \phi = V^\mu \partial_\mu \phi$$

$$\mathcal{L}_V B^\mu = V^\nu \partial_\nu B^\mu - B^\nu \partial_\nu V^\mu$$

$$= V^\nu \nabla_\nu B^\mu - B^\nu \nabla_\nu V^\mu$$

$$\mathcal{L}_V B_\mu = V^\nu \partial_\nu B_\mu + B^\nu \partial_\mu V_\nu$$

$$\mathcal{L}_V T_{\mu\nu} = V^\rho \partial_\rho T_{\mu\nu} + T_{\rho\nu} \partial^\rho V_\mu + T_{\mu\rho} \partial^\rho V_\nu$$

$$\mathcal{L}_V A^{\cdots\mu\cdots}{}_{\cdots\nu\cdots} = V^\rho \partial_\rho A - A^{\cdots\rho\cdots}{}_{\cdots\nu\cdots} \partial_\rho V^\mu$$

$$+ A^{\cdots\mu\cdots}{}_{\cdots\rho}$$

Note $$\nabla_\rho A^{\cdots\mu\cdots}{}_{\cdots\nu\cdots} = \partial_\rho A^{\cdots\mu\cdots}{}_{\cdots\nu\cdots} + \Gamma^\mu_{\rho\alpha} A^{\cdots\alpha\cdots}{}_{\cdots\nu\cdots}$$

$$- \Gamma^\alpha_{\rho\nu} A^{\cdots\mu\cdots}{}_{\cdots\alpha\cdots}$$

Apply to metric, ie. $T_{\mu\nu} = g_{\mu\nu}$

$$\mathcal{L}_V g_{\mu\nu} = V^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu V^\rho + g_{\mu\rho}\partial_\nu V^\rho$$
$$= V^\rho \partial_\rho g_{\mu\nu} + 2 g_{\rho(\mu}\partial_{\nu)} V^\rho$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\alpha_{\mu\nu} V_\alpha$$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left(2\partial_{(\mu} g_{\nu)\rho} - \partial_\rho g_{\mu\nu}\right)$$

$$\nabla_\mu V_\nu = \partial_\mu (g_{\nu\rho} V^\rho) - \Gamma^\nu_{\mu\alpha} V_\alpha$$
$$= V^\rho \partial_\mu g_{\nu\rho} + g_{\nu\rho}\partial_\mu V^\rho - \frac{1}{2} g^{\rho\alpha}(2\partial_{(\mu} g_{\nu)\rho} - \partial_\rho g_{\mu\nu}) V_\alpha$$

$$= \underline{V^\rho \partial_\mu g_{\nu\rho}} + g_{\nu\rho}\partial_\mu V^\rho - \underline{\frac{1}{2} V^\rho \partial_\mu g_{\nu\rho}} - \frac{1}{2} V^\rho \partial_\nu g_{\mu\rho}$$
$$+ \frac{1}{2} V^\rho \partial_\rho g_{\mu\nu}$$

$$= \frac{1}{2} V^\rho \partial_\rho g_{\mu\nu} + g_{\nu\rho} \partial_\mu V^\rho + \frac{1}{2} V^\rho \partial_\mu g_{\nu\rho} - \frac{1}{2} V^\rho \partial_\nu g_{\mu\rho}$$

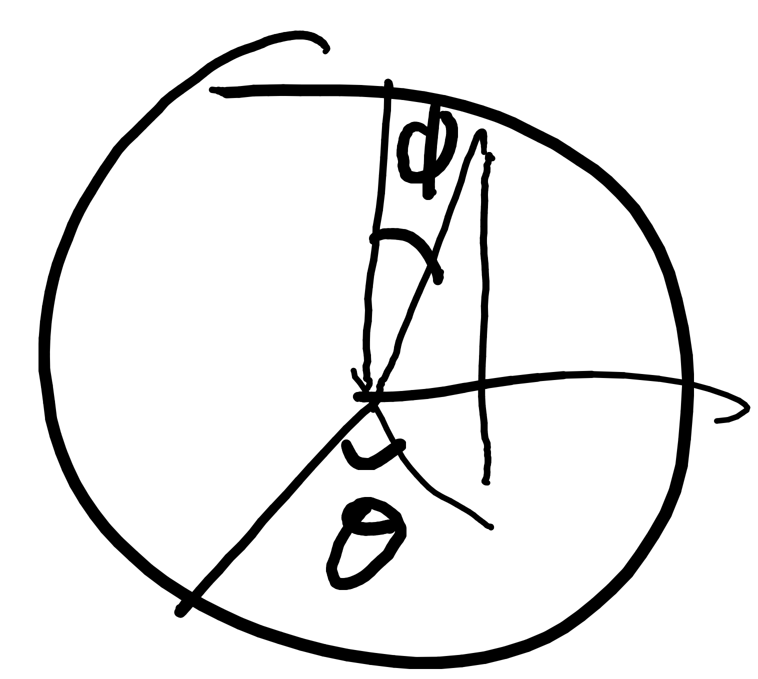
$$= \frac{1}{2} V^\rho \partial_\rho g_{\mu\nu} + g_{\nu\rho} \partial_\mu V^\rho + V^\rho \partial_{[\mu} g_{\nu]\rho}$$

$$\Rightarrow \nabla_{(\mu} V_{\nu)} = \frac{1}{2} V^\rho \partial_\rho g_{\mu\nu} + g_{\rho(\nu} \partial_{\mu)} V^\rho$$

$$= \frac{1}{2} \mathcal{L}_V g_{\mu\nu}$$

GR 21

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$



a) work in spherical coords

$$\vec{x} = (\overset{1}{\theta}, \overset{2}{\varphi}) \qquad \vec{x}_0 = (\theta_0, 0)$$

$$\vec{A}(\vec{x}_0) = \vec{e}_\theta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

From earlier assignment

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\varphi \qquad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta$$

$$= \frac{\cos\theta}{\sin\theta}$$

parallel transport along $\gamma : [0, 2\pi] \longrightarrow S^2$

$\lambda \mapsto \begin{pmatrix} \theta_0 \\ \varphi \end{pmatrix}$

$$D_\lambda A^i = D_\varphi A^i = \partial_\varphi A^i + \Gamma^i_{\varphi k} A^k$$

$$= \partial_\varphi A^i + \Gamma^i_{\varphi\theta} A^\theta + \Gamma^i_{\varphi\varphi} A^\varphi \overset{!}{=} 0$$

① $i = \theta$: $D_\varphi A = \partial_\varphi A^\theta + \Gamma^\theta_{\varphi\varphi} A^\varphi = \partial_\varphi A^\theta - \sin\theta\cos\theta A^\varphi$

② $i = \varphi$: $\partial_\varphi A^\varphi + \Gamma^\varphi_{\varphi\theta} A^\theta = \partial_\varphi A^\varphi + \cot\theta_0 A^\theta$

along curve $\theta = \theta_0$

∂_φ (I) : $\partial^2_\varphi A^\theta - \sin\theta\cos\theta\, \partial_\varphi A^\varphi = 0$

$$\overset{\text{II}}{=} \partial^2_\varphi A^\theta + \sin\theta\cos\theta\cot\theta A^\theta$$

$$= \partial^2_\varphi A^\theta + \cos^2\theta A^\theta = 0$$

$$A^{\theta}(\varphi) = A\cos(\cos\theta_0\,\varphi) + B\sin(\cos\theta_0\,\varphi)$$

We have initial cond $A^{\theta}(\varphi=0) = 1$

$\Rightarrow A = 0$

$$A^{\theta}(\varphi) = \cos(\cos\theta_0\,\varphi)$$

$$\Rightarrow \ldots \Rightarrow A^{\varphi} = -\frac{1}{\sin\theta_0}\sin(\cos\theta_0\,\varphi)$$

$$A^i(x_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow A^i(\theta_0, 2\pi) = \begin{pmatrix} \cos(2\pi\cos\theta_0) \\ -\frac{\sin(2\pi\cos\theta_0)}{\sin\theta_0} \end{pmatrix}$$

$\rightarrow$ not same vector!

$$\frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} = 0$$

$$\frac{d^2\theta}{d\lambda^2} - \left(\frac{d\varphi}{d\lambda}\right)^2 \sin\theta\cos\theta = 0$$

$$\frac{d^2\varphi}{d\lambda^2} + 2\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda}\cot\theta = 0$$

Lagrangian : $L = \frac{1}{2}\left(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2\right)$

$$\frac{\partial L}{\partial\theta} = \sin\theta\cos\theta\,\dot\varphi^2 \qquad \frac{\partial L}{\partial\varphi} = 0$$

$$\frac{\partial L}{\partial\dot\theta} = \dot\theta \qquad \frac{\partial L}{\partial\dot\varphi} = \sin\theta\dot\varphi^2$$

Euler - Lagrange

$L(x^i(\lambda), \dot{x}^i(\lambda)) \rightsquigarrow$ EOM at

$$\frac{\partial L}{\partial x^i} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^i} = 0$$

$$\Rightarrow \quad x^i = \theta \qquad \dot{\varphi}^2 \sin\theta\cos\theta - \frac{d}{d\lambda}\dot{\theta} = 0 \qquad \mathrm{I}$$

$$x^i = \varphi \qquad \frac{d}{d\lambda}\left(\sin\theta\,\dot{\varphi}^2\right)$$

$$= \cos\theta\,\dot{\theta}\,\dot{\varphi}^2 + 2\sin\theta\,\dot{\varphi}\,\ddot{\varphi} = 0$$

$$\Rightarrow \ddot{\varphi} + 2\frac{\sin\theta}{\cos\theta}\dot{\theta}\dot{\varphi} = 0 \qquad \mathrm{II}$$