GR 17

we have $g_{\mu\nu} g^{\nu\beta} = \delta_\mu^\beta$

write $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$ ($\delta g_{\mu\nu}$ small)

$$\delta(g_{\mu\nu} g^{\nu\beta}) = \delta(\underbrace{\delta_\mu^\beta}_{=cst}) = 0$$

$$\delta g_{\mu\nu} g^{\nu\beta} + g_{\mu\nu} \delta g^{\nu\beta} \overset{!}{=} 0$$

$$g_{\mu\nu} \delta g^{\nu\beta} = - g^{\nu\beta} \delta g_{\mu\nu}$$

$$g^{\mu\alpha} g_{\mu\nu} \delta g^{\nu\beta} = - g^{\mu\alpha} g^{\nu\beta} \delta g_{\mu\nu}$$

$$\delta_\nu^\alpha \delta g^{\nu\beta} = - g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$$

$$\delta g^{\mu\beta} = - g^{\mu\alpha} g^{\beta\nu} \delta g_{\alpha\beta}$$

GR18

Recall $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\beta} V^\beta$

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\beta} V^\beta$$

From last time

$$\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} = \Gamma^\alpha_{\mu\alpha}$$

therefore $\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\alpha} V^\alpha$

$$= \partial_\mu V^\mu + \frac{1}{\sqrt{g}}\left(\partial_\mu \sqrt{g}\right) V^\mu$$

$$\sqrt{|g|}\, \nabla_\mu V^\mu = \sqrt{|g|}\, \partial_\mu V^\mu + \partial_\mu\left(\sqrt{|g|}\right) V^\mu$$

$$= \partial_\mu\left(\sqrt{|g|}\, V^\mu\right)$$

GR 19

$$S = \frac{1}{2k} \int d^4x \sqrt{-g} \, (R + 2\Lambda)$$

$$\delta \sqrt{-g} = \frac{1}{2\sqrt{-g}} (-\delta g)$$

Using Jacobi' formula $\delta g = g g^{\mu\nu} \delta g_{\mu\nu}$

$$\delta \sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu} \delta g_{\mu\nu}$$

(Note $\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$

$$= -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu}$$

$$\delta S = \frac{1}{2k} \int d^4x \, \delta\left(\sqrt{-g}\,(R + 2\Lambda)\right)$$

$$= \frac{1}{2k} \int d^4x \left[(\delta\sqrt{-g})(R + 2\Lambda) + \sqrt{-g}\,(\delta R) \right.$$

Using $\delta R = \delta(g^{\mu\nu} R_{\mu\nu})$

$$\delta S = \frac{1}{2K}\int d^4x\,(-1)\left(\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}\,(R+\Lambda)\right)$$

$$+\frac{1}{2K}\int d^4x\,\sqrt{-g}\left(\delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right)$$

$$= \frac{1}{2K}\int \sqrt{-g}\,\delta g^{\mu\nu}\Big(\overbrace{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R}^{=G_{\mu\nu}} - g_{\mu\nu}\Lambda\Big)$$

$$+\frac{1}{2K}\int d^4x\,\sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}$$

One can prove $\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}$

→ the second term is a total derivative

$$\int d^4x\,\sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu} \sim \int d^4x\,\sqrt{-g}\,\nabla_\nu(g^{\cdot\cdot}\Gamma^{\cdot}_{\cdot\cdot})$$

$$\sim \int d^4x\,\partial_\nu(\sqrt{-g}\,g\Gamma^{\cdot}_{\cdot\cdot})$$

$$= 0$$

$$\rightarrow \delta S = 0 \Leftrightarrow G_{\mu\nu} - g_{\mu\nu}\Lambda = 0$$

GR 20

1) Einstein equation in vacuum

want to show

$$G_{\mu\nu} = 0 \Leftrightarrow R_{\mu\nu} = 0$$

$\Leftarrow$

$$R_{\mu\nu} = 0 \Rightarrow R = g^{\mu\nu} R_{\mu\nu} = 0$$

$$\Rightarrow G_{\mu\nu} = 0$$

$\Rightarrow$

$$G_{\mu\nu} = 0 \Rightarrow G_\mu^{\ \mu} = g^{\mu\nu} G_{\mu\nu} = 0$$

$$\Rightarrow R_\mu^{\ \mu} - \frac{1}{2} g_\mu^{\ \mu} R = 0$$

$$R - \frac{n}{2} R = 0 \Rightarrow R = 0$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$$

2) show $G_{\mu\nu} = 0 \overset{?}{\Leftrightarrow} S = 0$

$\underline{\Rightarrow}$ $G_{\mu\nu} = 0 \to G^{\mu}{}_{\mu} \sim R = 0$

$$S = \int d^m x \sqrt{-g} \underbrace{R}_{=0} = 0$$

$\underline{\Leftarrow}$

not correct, imagine (schematically)

$$R(-x) = -R(x)$$

$$S = S_{x>0} + S_{x<0} = S_{x>0} - \overbrace{S_{x>0}}^{x \to -x}$$

$= 0$