covariant derivatives

$$\nabla_\mu B^\nu = \partial_\mu B^\nu + \Gamma^\nu_{\mu\alpha} B^\alpha$$

Directional derivative: $\nabla_X B^\mu = X^\nu \nabla_\nu B^\mu$

Note: if $\phi(x)$ scalar field

$$\nabla_\mu \phi = \partial_\mu \phi \quad \text{(no index to contract}$$

∇_μ follow the product rule

$$\nabla_\mu (A^\alpha B_\beta) = (\nabla_\mu A^\alpha) B_\beta + A^\alpha \nabla_\mu B_\beta$$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} (\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu})$$

Uniqueness of Γ

Let $\nabla^{(\psi)}_\mu A^\rho = \partial_\mu A^\rho + \psi^\rho_{\mu\alpha} A^\alpha$

if we demand compatibility of ∇^ψ w/ $g_{\mu\nu}$

$$\nabla^{(\psi)}_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \psi^\alpha_{\mu\nu} g_{\alpha\rho} - \psi^\alpha_{\mu\rho} g_{\nu\alpha}$$
$$= \partial_\mu g_{\nu\rho} - \psi_{\mu\nu|\rho} - \psi_{\mu\rho|\nu}$$

if we assume $\psi^\alpha_{\mu\nu} = \psi^\alpha_{(\mu\nu)}$

$$0 = \nabla^{(\psi)}_\mu g_{\nu\lambda} + \nabla^{(\psi)}_\nu g_{\mu\lambda} - \nabla_\lambda g_{\mu\nu}$$
$$= \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}$$
$$- \psi_{\mu\nu|\lambda} - \cancel{\psi_{\mu\lambda|\nu}} - \psi_{\nu\mu|\lambda} - \cancel{\psi_{\nu\lambda|\mu}}$$
$$+ \cancel{\psi_{\lambda\mu|\nu}} + \cancel{\psi_{\lambda\nu|\mu}}$$

$$= 2(\Gamma_{\mu\nu|\lambda} - \psi_{\mu\nu|\lambda}) \overset{!}{=} 0$$

→ The unique connection that is compatible with metric is Γ

(in math literature, it is called the Levi-Civita connection)

connection w/ QFT

scalar field with $U(1)$

$$\phi(x) \to e^{i\alpha}\phi(x)$$

if $\alpha = \alpha(x)$

$$\partial_\mu \phi \to \partial_\mu(e^{i\alpha}\phi) = e^{i\alpha}\phi + i(\partial_\mu\alpha)\phi$$

gauge potential $A_\mu \to A_\mu - \partial_\mu \alpha$

gauge covariant derivative

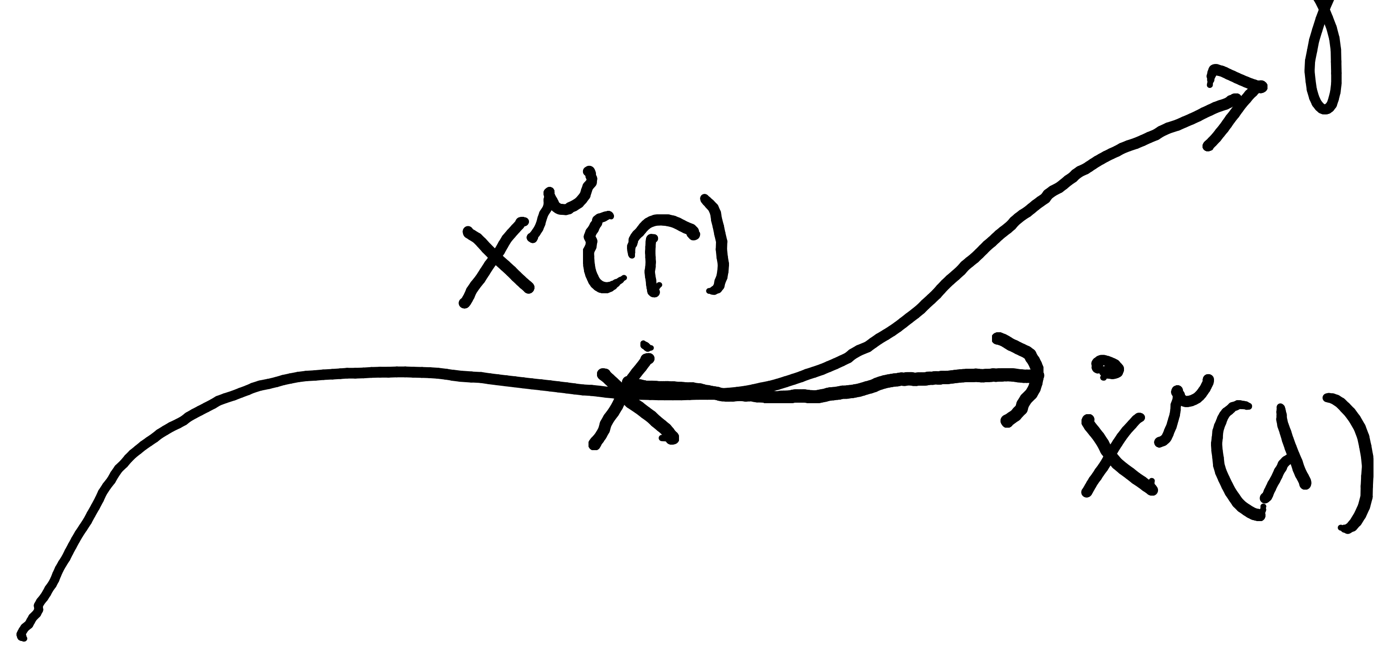
$$D_\mu \phi = \partial_\mu \phi + i A_\mu \phi$$

$$D_\mu \phi \to e^{i\alpha} D_\mu \phi$$

Can show $[D_\mu, D_\nu] = iF_{\mu\nu}$

	$U(1)$ EM	GR
connection	A_μ	$\Gamma_{\mu}{}^{\alpha}{}_{\nu}$
deriv.	$D = \partial + iA$	$\nabla = \partial + \Gamma$
curv	$[D_\mu, D_\nu]\phi = F_{\mu\nu}\phi$	$[\nabla_\mu, \nabla_\nu]V^\beta = R^\beta{}_{\alpha\mu\nu} V^\alpha$
name	$U(1)$ gauge theory	diffeomorphisms

Covariant derivative along curve γ



$$\gamma : [0,1] \to \mathcal{M}$$
$$\lambda \mapsto x^\mu(\lambda)$$

flat coord: $\dfrac{d}{d\lambda} = \dot{x}^\mu \partial_\mu \rightsquigarrow D_\lambda = \dfrac{D}{d\lambda} = \nabla_{\dot{x}} = \dot{x}^\mu \nabla_\mu$

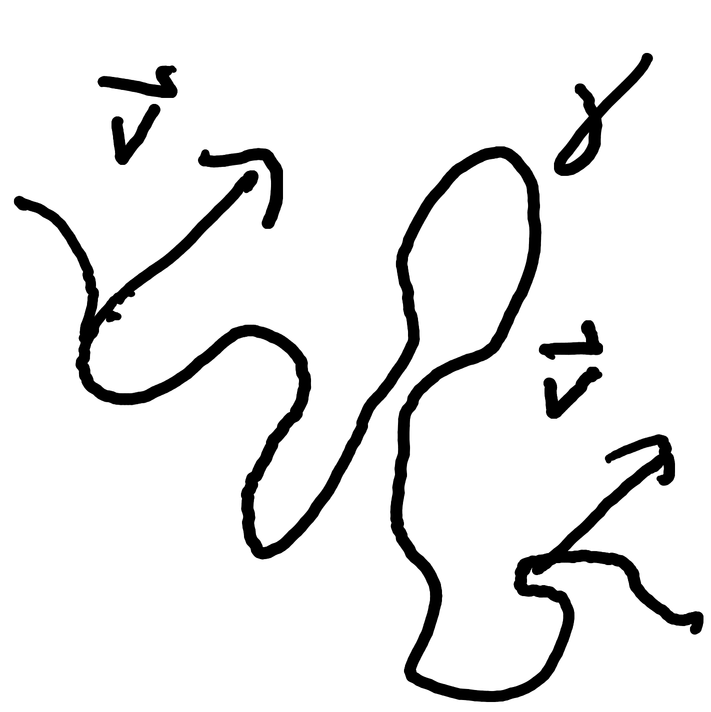
if $T^{\mu_1 \mu_2 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}$ is defined on curve

T is parallel transported along γ if

$$D\, T^{\cdots}{}_{\cdots} = 0$$

NB: since $\nabla_\nu g_{\alpha\beta} = 0$, metric is parallel transported along any curve

Euclidean



$\vec{v}$ is dragged parallelly along curve, ie

$\vec{v}(x(\lambda)) = \vec{v} = \overrightarrow{const} \Rightarrow \dfrac{d\vec{v}}{d\lambda} = 0$

curved: same but take curvature into account

$D_\lambda V^\mu = 0$

defines GR version of trajectory quantities

$x^\mu(\tau)$ position on worldline

$U^\mu = \dot{x}^\mu$ velocity " "

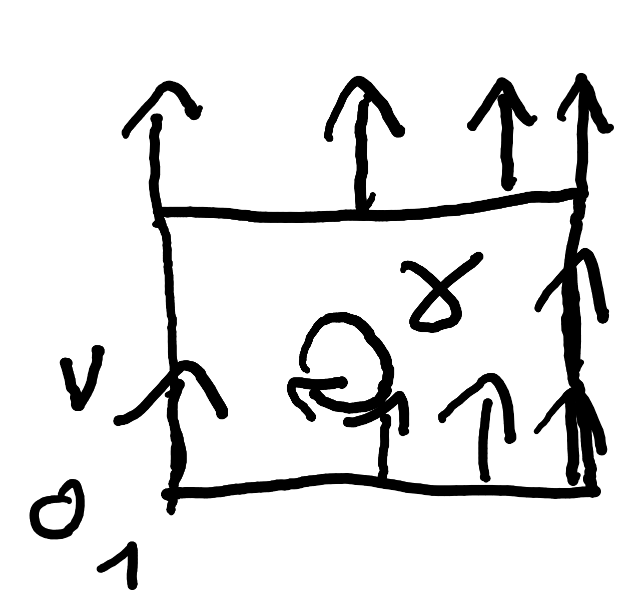
$$a^\mu = D_\tau \dot{x}^\mu = \ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho \quad (= U^\nu \nabla_\nu U^\mu)$$

free-falling observer

$a^\mu = 0 \iff$ geodesic

parallel transport and curvature (Bla 11.1)

take loop in flat space | curved space



$\vec{V}(0) = \vec{V}(1)$

$V_\mu(0) - V_\nu(1) \neq 0$

Using that along curve

$$0 = D_\tau V_\nu = \nabla_{\dot{x}} V_\nu = \dot{x}^\mu \partial_\mu V_\nu - \Gamma^\alpha_{\mu\nu} V_\alpha \dot{x}^\mu$$

$$\frac{d}{d\tau} V_\nu = \Gamma_{\mu\nu}^{\alpha} V_\alpha \dot{x}^\mu$$

$$\Rightarrow \quad V_\nu(\tau) = V_\nu(0) + \int_0^\tau d\lambda \; \Gamma^\alpha_{\mu\nu}(x(\lambda))\, \dot{x}(\lambda)\, V_\alpha(\lambda)$$

at leading order

$$V_\nu(\tau) - V_\nu(0) \sim -\frac{1}{2} \oint dx^\nu x^\rho R_{\mu\cancel{\alpha}\rho\nu} V^\alpha$$

GR 14

$$S = \int d\lambda \left(g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)$$

$$S(x+\varepsilon\delta x, \dot{x}+\varepsilon\delta\dot{x}) = S + \delta S \varepsilon + \dots$$

is stat if $\delta S = 0$

$$y^\mu = x^\mu + \delta x^\mu$$

$$g_{\mu\nu}(x+\delta x^\nu) = g_{\mu\nu} + \partial_\rho g_{\mu\nu} \delta x^\rho \qquad \delta\dot{x} = \frac{d}{d\tau}\delta x$$

$$(g_{\mu\nu} + \partial_\rho g_{\mu\nu} \delta x^\rho)(\dot{x}^\mu + \delta\dot{x}^\mu)(\dot{x}^\nu + \delta\dot{x}^\nu)$$

$$g(y)\dot{y}\dot{y} = 2 g_{\mu\nu} \dot{x}^\mu \delta\dot{x}^\nu + \partial_\rho g_{\mu\nu} \delta x^\rho \dot{x}^\mu \dot{x}^\nu$$

$$S = \int d\lambda \left(-2 \frac{d}{d\lambda}(g_{\mu\nu}\dot{x}^\mu)\delta x^\nu + \partial_\nu g_{\mu\rho} \dot{x}^\mu \dot{x}^\rho \delta x^\nu \right.$$

$$= \int \left(-2 g^{\nu\alpha} \partial_\rho g_{\mu\alpha} \dot{x}^\rho \dot{x}^\mu + \partial^\nu g_{\mu\rho} \dot{x}^\mu \dot{x}^\rho - 2\ddot{x}^\nu \right) \delta x_\nu$$

$$= \int -2(\ddot{x}^\nu + \Gamma^\nu_{\mu\rho} \dot{x}^\mu \dot{x}^\rho) = 0$$

GR 15 (Friday)

Brut force

Write $R_{\mu\nu\rho\sigma} = A_{\mu\nu\rho\sigma} + B_{\mu\nu\rho\sigma}$

using sym of $g_{\mu\nu} = g_{\nu\mu}$ and $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$

$$A_{\mu\nu\rho\sigma} = \cancel{g_{\mu\sigma,\nu\rho}} - \cancel{g_{\mu\rho,\sigma\nu}} + \cancel{g_{\nu\rho,\sigma\mu}} - g_{\sigma\nu,\rho\mu}$$

$$A_{\mu\sigma\rho\nu} = \cancel{g_{\mu\rho,\sigma\nu}} - g_{\mu\nu,\rho\sigma} + \cancel{g_{\sigma\nu,\rho\mu}} - \boxed{g_{\rho\sigma,\nu\mu}}$$

$$A_{\mu\nu\sigma\rho} = g_{\mu\nu,\rho\sigma} - \cancel{g_{\mu\sigma,\nu\rho}} + \boxed{g_{\rho\sigma,\nu\mu}} - \cancel{g_{\nu\rho,\sigma\mu}}$$

$$A_{\mu\nu\rho\sigma} + A_{\mu\sigma\nu\rho} + A_{\mu\rho\sigma\nu} = 0$$

$$B_{\mu\nu\rho\sigma} = g_{\tau\lambda}\left(\cancel{\Gamma^{\tau}_{\mu\sigma}\Gamma^{\lambda}_{\nu\rho}} - \cancel{\Gamma^{\tau}_{\mu\rho}\Gamma^{\lambda}_{\sigma\nu}}\right)$$

$$B_{\mu\sigma\nu\rho} = g_{\tau\lambda}\left(\cancel{\Gamma^{\tau}_{\mu\rho}\Gamma^{\lambda}_{\sigma\nu}} - \Gamma^{\tau}_{\mu\nu}\Gamma^{\lambda}_{\rho\sigma}\right)$$

$$B_{\mu\rho\sigma\nu} = g_{\tau\lambda}\left(\Gamma^{\tau}_{\mu\nu}\Gamma^{\lambda}_{\rho\sigma} - \cancel{\Gamma^{\tau}_{\mu\sigma}\Gamma^{\lambda}_{\nu\rho}}\right)$$

$$B_{\mu\nu\rho\sigma} + (\sigma\nu\rho) + (\rho\sigma\nu) = 0$$

→ Bianchi

2)

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)$$

$$\Gamma^{\rho}_{\mu\rho} = \frac{1}{2} g^{\rho\alpha}\left(\partial_\mu g_{\alpha\rho} + \cancel{\partial_\rho g_{\mu\alpha}} - \cancel{\partial_\alpha g_{\mu\rho}}\right)$$

$$= \frac{1}{2} g^{\rho\alpha}\partial_\mu g_{\alpha\rho}$$

Use Jacobi's formula

for matrix $A = A(t)$

$$\frac{d}{dt}\det(A) = (\det A)\,\mathrm{tr}\left(A^{-1}\frac{dA}{dt}\right)$$

(derivation on wikipedia

In components

$$\frac{1}{\det A}\frac{d}{dt}\det A = tr\left((A^{-1})^{ij}\frac{dA_{jk}}{dt}\right)$$

$$\frac{d}{dt}\ln\det A = (A^{-1})^{ij}\frac{dA_{ji}}{dt}$$

Applied to metric

$$\frac{\partial}{\partial x^\rho}\ln\det(g) = g^{\mu\nu}\frac{\partial g_{\nu\mu}}{\partial x^\rho}$$

$$\partial_\rho \det(g) = \det(g)\, g^{\mu\nu}\partial_\rho g_{\mu\nu}$$

writing $g = \det g$

$$\circledast\ \partial_\rho\sqrt{g} = \frac{1}{2\sqrt{g}}\partial_\rho g = \frac{1}{2\sqrt{g}}(g\, g^{\mu\nu}\partial_\rho g_{\mu\nu} = \frac{1}{2}\sqrt{g}\, g^{\mu\nu}\partial_\rho g_{\mu\nu}$$

$$\rightarrow \Gamma^{\rho}_{\mu\rho} = \frac{1}{2}g^{\rho\alpha}\partial_\mu g_{\rho\alpha} = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}$$