Recap

A **manifold** (=mfld) $\mathcal{M}$ a space that locally looks like $\mathbb{R}^d$. We say that

$d = \dim(\mathcal{M})$ is the dimension of $\mathcal{M}$

$\longrightarrow$ very informal definition

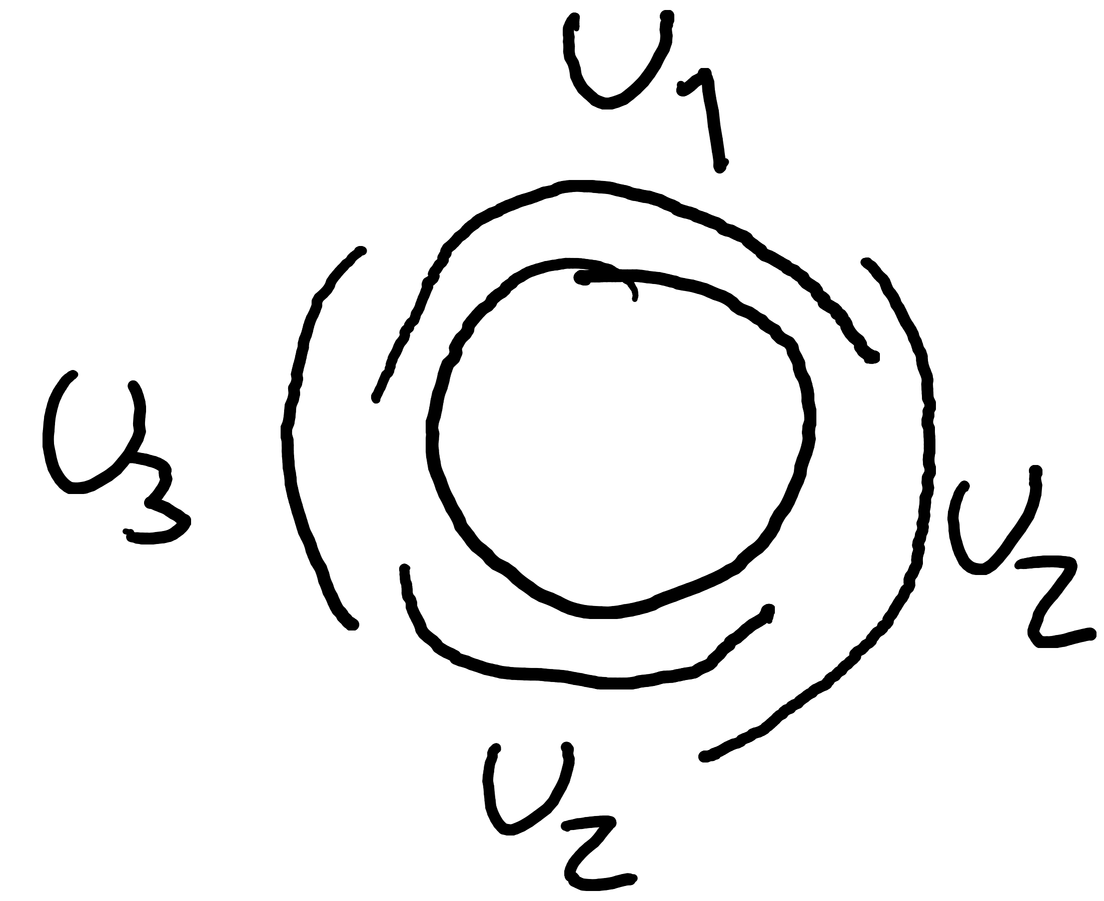
locally looks like $\mathbb{R}^d$: $\exists\ U_i \subset \mathcal{M}$, $\phi_i : U \to \mathbb{R}^d$ invertible

pairs (U_i, ϕ_i) is a <u>chart</u>

The collection of all charts = Atlas

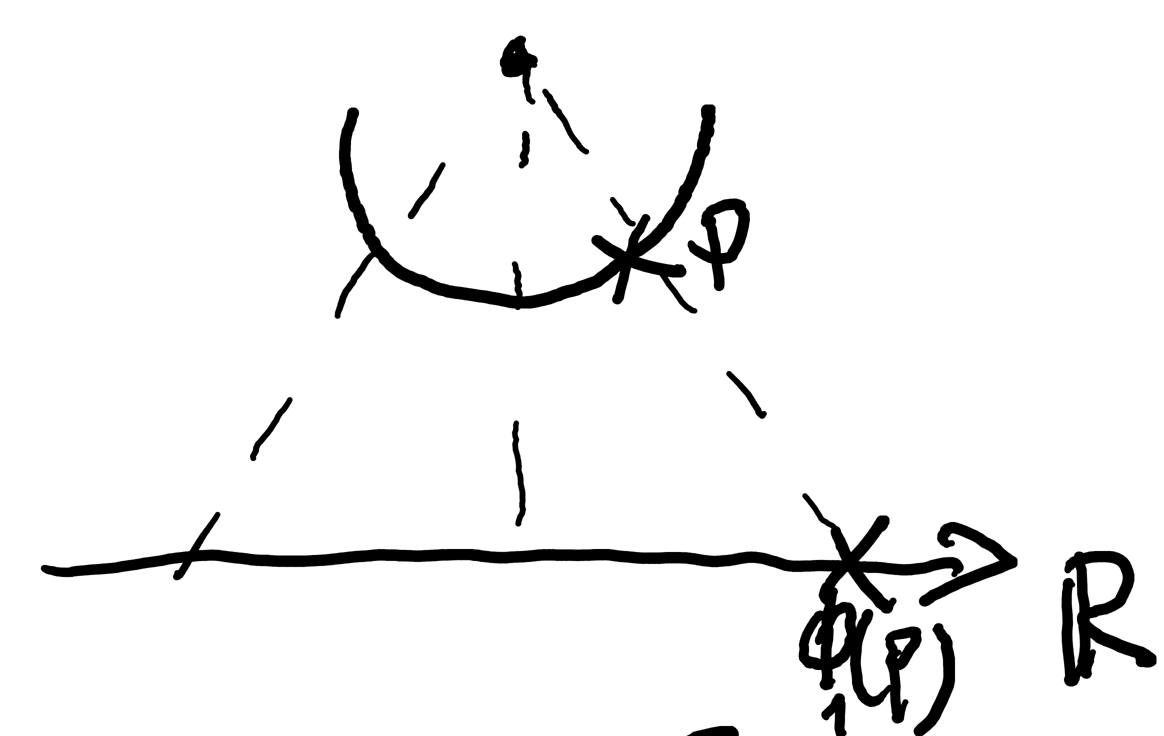
$$\mathcal{M} = \bigcup_i U_i$$

Example: circle S^1



$$S^1 = U_1 \cup U_2 \cup U_3 \cup U_4$$

$$\Phi_1 : U_1 \to \mathbb{R}$$

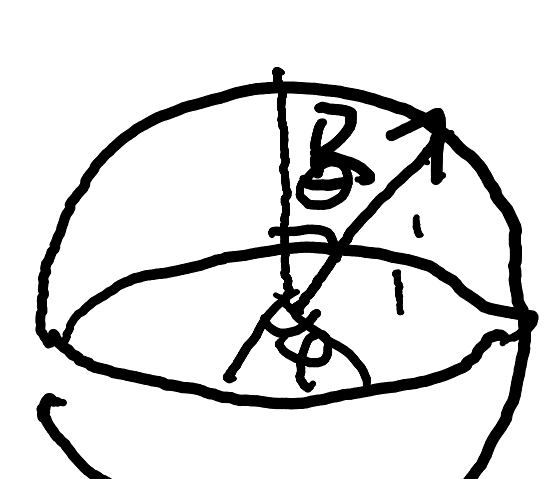


can be more formal using $x^2 + y^2 = 1$

if a mfld is endowed w/ metric $g_{\mu\nu}$, it is <u>Riemannian</u> (most of the time, mfld is assumed Riemannian)

metric: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

2-sphere



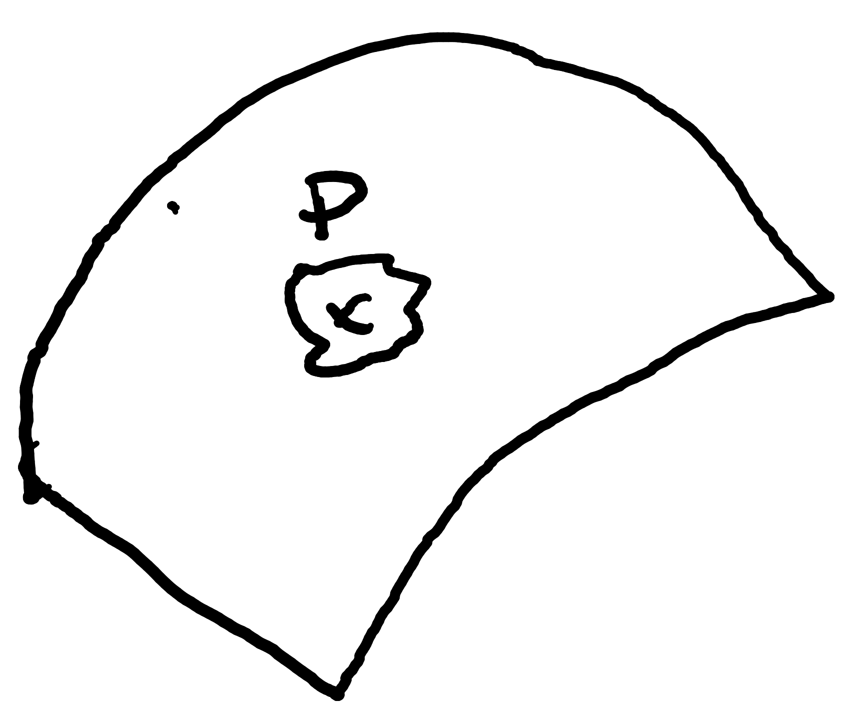
$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

having metric, can define distances

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

local coordinate

in a neighbourhood of a point, can always find a flat system of coordinates



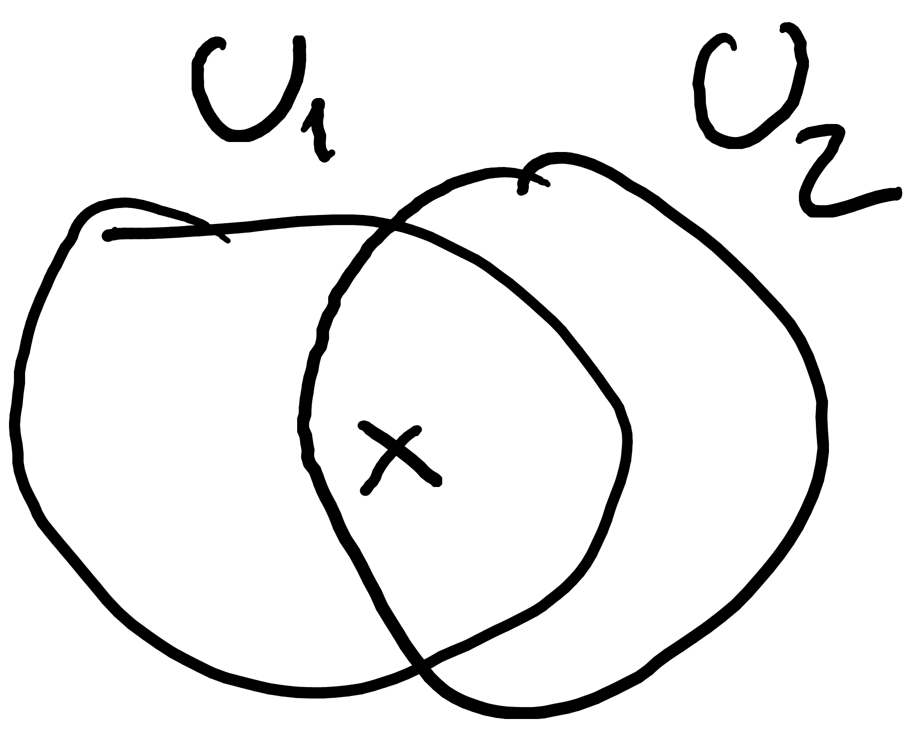
near p:

$$ds^2 \sim \eta_{\mu\nu}\, dx^\mu dx^\nu \quad \text{Lorentzian}$$

$$ds^2 \sim \delta_{\mu\nu}\, dx^\mu dx^\nu \quad \text{Euclidean}$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \delta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

change of coordinate = change of chart



$\phi_1 : U_1 \longrightarrow \mathbb{R}^d \qquad x^\mu$

$\phi_2 : U_2 \longrightarrow \mathbb{R}^d \qquad \tilde{x}^\mu$

on $U_1 \cap U_2$: $\phi_2 \circ \phi_1^{-1} : x^\mu \rightarrow \tilde{x}^\mu$

<u>worldline:</u> $\gamma \subset \mathcal{M}$ such that $\dim(\gamma) = 1$

$\dim(\gamma) = 1 \Rightarrow 1$ coordinate σ

metric: we can measure distances

$$L(P_1, P_2) = \int_{P_1}^{P_2} ds = \int_{P_1}^{P_2} \sqrt{g(\sigma)}\, d\sigma$$

minimal path = EoM of L = geodesic

$$\frac{dx^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma} = 0$$

Christoffel

<u>equivalence principle</u>: free-falling observers follow a geodesic in $\mathcal{M}$

GR: $\mathcal{M}$ is 4D Lorentzian mfld

in lecture: 1) rest frame $\frac{d\xi^\mu}{d\tau} = 0$

2) $\xi^\mu \longrightarrow x^\mu$ arbitrary

3) use chain rule

<u>tensors</u>: A tensor $T^{\mu\nu}{}_{\rho\sigma}$ transform under general coord transfo

$$x^\mu \longrightarrow \tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} x^\alpha$$

$$T^{\mu\nu}{}_{\rho\sigma} \longrightarrow \tilde{T}^{\mu\nu}{}_{\rho\sigma} = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\frac{\partial x^\gamma}{\partial \tilde{x}^\rho}\frac{\partial x^\delta}{\partial \tilde{x}^\sigma} T^{\alpha\beta}{}_{\gamma\delta}$$

NB: Γ is <u>not</u> a tensor $\tilde{\Gamma}^\mu_{\nu\rho} = \underbrace{\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\frac{\partial x^\gamma}{\partial \tilde{x}^\omega}\Gamma^\alpha_{\beta\gamma}}_{\text{cov}} + \underbrace{\frac{\partial^2 x^\sigma}{\partial \tilde{x}^\omega \partial \tilde{x}^\rho}\frac{\partial \tilde{x}^\mu}{\partial x^\sigma}}_{\text{non cov!}}$$

GR "=" chain rules on $\mathcal{M}$

GR7

$$[\phi_a, \phi_b] \longrightarrow \mathcal{M}$$

$$\phi \longrightarrow x^\mu(\phi)$$

a) $$\eta_{\mu\nu} \frac{dx^\mu}{d\phi} \frac{dx^\nu}{d\phi} \quad \begin{array}{ll} > 0 & \text{time} \\ = 0 & \text{light} \\ < & \text{space} \end{array}$$

under reparametrisation

$$\phi \rightarrow \varphi(\phi)$$

$$x^\mu(\varphi) = \frac{dx^\mu}{d\varphi} \frac{d\varphi}{d\phi}$$

$$\Rightarrow \eta_{\mu\nu} \frac{dx^\mu}{d\varphi} \frac{d\varphi}{d\phi} \frac{dx^\nu}{d\varphi} \frac{d\varphi}{d\phi} = \eta_{\mu\nu} \frac{dx^\mu}{d\varphi} \frac{dx^\nu}{d\varphi} \underbrace{\left(\frac{d\varphi}{d\phi}\right)^2}_{>0}$$

$\boxed{c=1}$

b)

$$T = \int d\tau = \int ds/c$$

$$= \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\phi} \frac{dx^\nu}{d\phi}}\, d\phi$$

$$\int d\psi \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\psi} \frac{dx^\nu}{d\psi}} = \int \underbrace{d\psi \;\; \frac{d\phi}{d\psi}}_{= d\phi} \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\phi} \frac{dx^\nu}{d\phi}} = T$$

c) $\psi \overset{\text{set}}{=} \pi - \arccos\phi \Leftrightarrow \phi = \cos(\pi - \psi) = -\cos\psi$

$$x^0 = 2(\pi - \arccos\phi) = 2\psi$$

$$x^1 = \phi = -\cos\psi$$

$$x^2 = \sqrt{1-\phi^2} = \sqrt{1-\cos^2\psi} = \sin\psi$$

$$x^3 = 0 = 0$$

$$\frac{dx^\mu}{d\phi} = \begin{pmatrix} 2 \\ \sin\psi \\ \cos\psi \\ 0 \end{pmatrix} \qquad \begin{matrix} \phi_a = -1 \\ \phi_b = 1 \end{matrix} \Leftrightarrow \begin{matrix} \psi_a = 0 \\ \psi_b = \pi \end{matrix}$$

$$T = \int_{\varphi_2}^{\varphi_3} \sqrt{\eta_{\mu\nu} \frac{dx^{\mu}}{d\varphi} \frac{dx^{\nu}}{d\varphi}}$$

$$= \int_0^{\pi} \sqrt{4 - \sin^2\varphi - \cos^2\varphi}\, d\varphi = \int_0^{\pi} \sqrt{3} = \sqrt{3}\pi$$

GR8

$$S^{\mu_1}{}_{\mu_2 \cdots \mu_n} = g^{\mu_1 \nu} T_{\nu \mu_2 \cdots \mu_n}$$

$$g_{\mu_1 \alpha} S^{\alpha}{}_{\mu_2 \cdots \mu_n} = g_{\alpha \mu_1} g^{\alpha \nu} T_{\nu \mu_2 \cdots \mu_n}$$

$$= \delta^{\nu}_{\mu_1} T_{\nu \mu_2 \cdots \mu_n}$$

$$= T_{\mu_1 \cdots \mu_n}$$

GR9

$$g^{\mu\nu} = \frac{\partial x^\mu}{\partial \xi^\rho}\frac{\partial x^\nu}{\partial \xi^\sigma}\eta^{\rho\sigma}$$

in rest frame where $x^\nu = \xi^\nu$

$$\eta_{\mu\nu}\eta^{\nu\alpha} = \delta^\alpha_\mu$$

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

$$g^{\mu\nu}g_{\mu\nu} = \left(\frac{\partial x^\mu}{\partial \xi^\rho}\frac{\partial x^\nu}{\partial \xi^\sigma}\eta^{\rho\sigma}\right)\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}\right)$$

$$\frac{\partial x^\mu}{\partial \xi^\rho}\frac{\partial x^\rho}{\partial x^\alpha} = \frac{\partial x^\mu}{\partial x^\alpha} = \delta^\mu_\alpha$$

$$g^{\mu\nu}g_{\mu\nu} = \eta^{\rho\sigma}\delta^\alpha_\rho\delta^\beta_\sigma\eta_{\alpha\beta} = \eta^{\rho\alpha}\eta_{\alpha\sigma} = \delta^\rho_\sigma$$