Linearised gravity $\qquad \eta = \begin{pmatrix} 1 & \\ & -\mathbb{1} \end{pmatrix}$

define $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)$

$\to \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

raise indices w/ η

$$R_{\mu\nu} \sim \frac{1}{2}\left(\partial_\alpha \partial_\nu h^\alpha_\mu + \partial_\alpha \partial_\mu h^\alpha_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}\right)$$

$$= \frac{1}{2}\left(\partial_\nu D_\mu + \partial_\mu D_\nu - \Box h_{\mu\nu}\right)$$

$$D_\nu = \partial_\alpha h^\alpha{}_\nu - \frac{1}{2}\partial_\nu h \quad ; \quad h = h^\mu_\mu = \eta^{\mu\nu} h_{\mu\nu}$$

diff : $x^\mu \longrightarrow x^\mu + \varepsilon^\mu$

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\varepsilon g_{\mu\nu}$$

$$h_{\mu\nu} \to h_{\mu\nu} + \mathcal{L}_\varepsilon \eta_{\mu\nu}$$

$$= h_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$$

choose ε_ν such that $D_\nu = 0$

(de Donder) gauge

ex: $D_\mu = 0 \Leftrightarrow g^{\mu\nu} \Gamma^\rho_{\mu\nu} = 0$

in that gauge

$$R_{\mu\nu} = -\frac{1}{2} \Box h_{\mu\nu}$$

$\rightarrow$ Einstein equation in vacuum

$$R_{\mu\nu} = 0 \Leftrightarrow \Box h_{\mu\nu} = 0$$

solution: $h_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x^\alpha}$ $k^2 = 0$

de Donder: $D_\mu = 0 \Rightarrow k^\nu \varepsilon_{\mu\nu} = 0$

choose $\varepsilon_\mu = a_\mu e^{ik_\alpha x^\alpha}$

$$A_{\mu\nu} \longrightarrow A_{\mu\nu} + i(k_\mu a_\nu + k_\nu a_\mu)$$

choose a such that $\varepsilon_{\mu 0} = \varepsilon^\mu_\mu = 0$

$$h^+_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad h^\times_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$h_{\mu\nu} = (A^+ h^+_{\mu\nu} + A^\times h^\times_{\mu\nu}) e^{ik_\alpha x^\alpha}$$

$$k^\mu = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix}$$

$$x^\mu \longrightarrow \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu \qquad x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x)$$

$$= (\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu \, g_{\rho\sigma}(x)$$

at linearized level, invariant if

$$(\Lambda^{-1})^\rho{}_\nu \, \eta_{\rho\sigma} (\Lambda^{-1})^\sigma{}_\mu = \eta \implies \text{Lorentz transformation}$$

3 rot + 3 boosts

$$\tilde{h}_{\mu\nu}(\tilde{x}) = (\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu \, h_{\rho\sigma}(x = \Lambda^{-1}\tilde{x})$$

a) rotations

as matrix $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ such that

$R^t = R$. Along Θ (= around z)

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & +\sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda^t \eta \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \eta$$

$$\Lambda^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & R^t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

if $k^\mu = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix} \rightarrow \Lambda k = k$

$$\tilde{h}_{\mu\nu} = \Lambda^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^+ & A^\times & 0 \\ 0 & A^\times & -A^+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Lambda \; \underbrace{e^{i k_\alpha (\Lambda^{-1} \tilde{x})^\alpha}}_{\substack{= e^{i (\Lambda k)_\alpha \tilde{x}^\alpha} \\ = e^{i k_\alpha \tilde{x}^\alpha}}}$$

$$\begin{pmatrix} \tilde{A}^+ \\ \tilde{A}^\times \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \begin{pmatrix} A^+ \\ A^\times \end{pmatrix}$$

b) boost along z w/ rapidity $\beta = \operatorname{atan}(v)$

$$\Lambda = \begin{pmatrix} \cosh\beta & 0 & 0 & -\sinh\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sinh\beta & 0 & 0 & \cosh\beta \end{pmatrix}$$

$\tilde{A}_\mu = A_\mu$ clearly, on the other hand

$$e^{i k_\alpha (\Lambda^{-1} \tilde{x})} = e^{i (\Lambda k)_\alpha \tilde{x}^\alpha}$$

$$\Lambda^\mu{}_\nu k^\nu = \Lambda^\mu{}_\nu \begin{pmatrix} \omega \\ 0 \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} \omega \cosh\beta - k \sinh\beta \\ 0 \\ 0 \\ k \cosh\beta - \omega \sinh\beta \end{pmatrix}$$

w/ $\omega = k$

rotations $\longrightarrow$ change amplitude

boost $\longrightarrow$ shifts frequency

2)

show $g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0 \Leftrightarrow D_{\mu} = 0$ at

linear order

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = g^{\mu\nu}\frac{1}{2}g^{\rho\alpha}(\partial_{\mu}g_{\nu\alpha} + \partial_{\nu}g_{\mu\alpha} - \partial_{\alpha}g_{\mu\nu})$$

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}(\eta^{\mu\nu} - h^{\mu\nu})(\eta^{\rho\alpha} - h^{\rho\alpha})(\partial_{\mu}h_{\nu\alpha} + \partial_{\nu}h_{\mu\alpha} - \partial_{\alpha}h_{\mu\nu})$$

$$= \frac{1}{2}(\eta^{\mu\nu}\eta^{\rho\alpha})(\partial_{\mu}h_{\nu\alpha} + \partial_{\nu}h_{\mu\alpha} - \partial_{\alpha}h_{\mu\nu}) + O(h^2)$$

$$= \frac{1}{2}(\partial^{\nu}h_{\nu}^{\rho} + \partial^{\mu}h_{\mu}^{\rho} - \partial^{\rho}h) + O(h^2)$$

$$= \partial^{\nu}h_{\nu}^{\rho} - \partial^{\rho}h + O(h^2)$$

$$= D^{\rho} + O(h^2)$$